Anosov-representations: Basic definitions and properties

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Abstract

These informal notes are a brief summary of the talk I gave, with the same title, at the Workshop on Higher Teichmüller-Thurston theory, which took place in Northport Maine in the last week of June 2013. Use at your own risk.

1 Informal introduction

Anosov representations were introduced by Labourie [Lab] and describe so-called 'Anosov structures'. The basic idea is that, rather than encoding a sort of geometric structure on your manifold, such representations encode a type of dynamical structure that is analogous to that of an Anosov flow. This turns out to be a more flexible type of structure that is applicable in a wider context.

After defining the concept, Labourie went on to establish various results about Anosov representations. In particular, he showed that representations in Hitchin components are Anosov and used this to extend the known results about Hitchin components. For example, he showed that the mapping class group acts property on the Hitchin component, and that all representations in the Hitchin component are discrete, faithful, and purely loxodromic.

Taking the idea behind Anosov representations further, Guichard and Wienhard [GW] expanded the notion from Anosov structures on manifolds to representations of arbitrary word hyperbolic groups and showed that the definition may be reformulated in terms of a pair of *Anosov maps* from the boundary of the word hyperbolic group into certain compact homogeneous spaces. Working in this framework, they showed, among other things, that the set of Anosov representations is open in the representation variety.

2 Motivation

Upon a first encounter with the topic, the definition of Anosov representations can appear quite opaque making it difficult to perceive the idea behind the concept. In attempts to circumvent this frustration, in this section I'll discuss a motivational picture that I hope will help illuminate the definition of Anosov representations in §3.

2.1 Anosov flows

Before getting into the specifics of Anosov representations, let us first recall the motivating picture of Anosov flows. For a more detailed discussion of Anosov flows and their properties, see the article [Pla] by Plante and the references therein.

Let M be a compact Riemannian manifold equipped with a C^1–flow $\phi_t\colon M \to M$. This is said to be an **Anosov flow** if there is a ϕ_t–invariant splitting

$$TM = E^s \oplus E^u \oplus E^T$$

of the tangent bundle TM into subbundles E^s, E^u, and E^T (so each bundle must be invariant under the derivative $D\phi_t$) that satisfy

1. E^T is a line bundle that is tangent to the flow ϕ_t,
2. E^u is *expanding*, that is, there exist constants $A > 0$, $\mu > 1$ so that for all $t \in \mathbb{R}$ and $v \in E^u$ we have
$$\|D\phi_t(v)\| \geq A\mu^t \|v\|,$$
3. E^s is *contracting*, that is, there exist constants $B > 0$, $\lambda < 1$ so that for all $t \in \mathbb{R}$ and $v \in E^s$ we have
$$\|D\phi_t(v)\| \leq B\lambda^t \|v\|.$$

(Above, the norms of vectors are calculated with respect to the Riemannian metric.) We remark that since M is compact, conditions (2) and (3) are independent of the chosen metric; thus being Anosov is truly a property of the flow itself.

In the case of an Anosov flow, it is known that the bundles E^u and E^s are in fact integrable, meaning that they are tangent to the leaves of C^1 foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ of M.

Example 2.1. The canonical example of an Anosov flow is the geodesic flow on (the unit tangent bundle of) a compact negatively curved Riemannian manifold. To briefly see why such a flow is Anosov, let us consider the case of a closed surface S equipped with a hyperbolic metric. Then the universal cover $\tilde{S}$ is identified with the Poincaré upper half plane $\mathbb{H}^2$ and the unit tangent bundle $T^1(\tilde{S})$ with $\mathrm{PSL}(2,\mathbb{R})$. For concreteness, let's identify $I \in \mathrm{PSL}(2,\mathbb{R})$ with the vertical tangent vector at the point $i \in \mathbb{H}^2$.

Now, the tangent space $T_I(\mathrm{PSL}(2,\mathbb{R}))$ is the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of 2×2 traceless matrices, which has a basis given by
$$G_I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \quad X_I = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \quad Y_I = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right).$$
These generate left-invariant vector fields G, X, and Y on $\mathrm{PSL}(2,\mathbb{R})$ (which thus descend to vector fields on $T^1(S)$) which induce a splitting of $T(\mathrm{PSL}(2,\mathbb{R}))$ as the sum of 3 line bundles E_G, E_X, and E_Y.

Exponentiating in the direction of these basis vectors gives the matrices
$$g_t := \exp(tG_I) = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}, \quad x_t := \exp(tX_I) = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right), \quad y_t := \exp(tY_I) = \left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right).$$
Multiplication on the *right* by g_t defines a flow on $\mathrm{PSL}(2,\mathbb{R})$ that is parallel to the vector field G (and similarly for x_t and y_t); in fact g_t is exactly the geodesic flow on $T^1(\mathbb{H}^2)$, and x_t and y_t are the horocycle flows. Since these flows commute with the left action on $\pi_1(S)$ on $\mathrm{PSL}(2,\mathbb{R})$, they descend to flows on $T^1(S)$.

With this concrete description of g_t, it is easy to check that the geodesic flow is Anosov with respect to the splitting $T(\mathrm{PSL}(2,\mathbb{R})) = E_G \oplus E_X \oplus E_Y$. Indeed, E_G is tangent to the flow, and the facts that $x_s g_t = e^{-t} g_t x_s$ and $y_s g_t = e^t g_t y_s$ show that the flow g_t is contracting and expanding on E_X and E_Y, respectively.

2.2 Philosophical ramblings

When studying representation spaces $\mathrm{Hom}(\Gamma, G)$, a goal is often to understand what sort of information these representations encode. For example, when $\Gamma = \pi_1(S)$ for a closed surface S and $G = \mathrm{PSL}(2,\mathbb{R})$, the discrete faithful representations in $\mathrm{Hom}(\Gamma, G)$ exactly encode hyperbolic structures on S. The question then arises, what sort of structure may be encoded by a representation into, for example, a larger Lie group?

We have seen in Example 2.1 that if a representation $\pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ defines a hyperbolic structure on S, then via the geodesic flow on $T^1(S)$ we actually have the extra structure of an Anosov system. In fact, as Example 2.1 illustrated, the distributions comprising the Anosov system in fact came from the Lie group $\mathrm{PSL}(2,\mathbb{R})$ itself. Now, if we change $\mathrm{PSL}(2,\mathbb{R})$ into a larger-dimensional Lie group G, it becomes quite difficult for a representation $\pi_1(S) \to G$ to encode a geometric structure on S—any 'developing map' from $\tilde{S}$ to the symmetric space G/K could not be a local homeomorphism because dimensions would not match up, and so we are left searching for other candidate G–spaces to model the desired geometry on S.

Labourie's insight in [Lab] was that, in spite of these difficulties, perhaps such a representation $\pi_1(S) \to G$ could still encode some sort of *dynamical* structure on S in the same way that we saw the structure of the Lie group $\mathrm{PSL}(2,\mathbb{R})$ lead to an Anosov flow on $T^1(S)$ in Example 2.1. This idea turned out to work, and the added dynamical structure allowed Labourie and others to prove new results about the representation spaces $\mathrm{Hom}(\pi_1(S), G)$.

3 Anosov representations

Let G be a semisimple Lie group, and let (P^+, P^-) be a pair of opposite parabolic subgroups (see the notes [Col] from Brian Collier's talk on semisimple Lie groups for the definition and properties of parabolic subgroups). Then the quotients $\mathcal{F}^\pm = G/P^\pm$ are compact homogeneous spaces (since $P^\pm$ are parabolic). We also consider the homogeneous space $\mathcal{X} = G/L$, where $L = P^+ \cap P^-$.

If we let G act diagonally on $\mathcal{F}^+ \times \mathcal{F}^-$, then L is exactly the stabilizer of (eP^+, eP^-) and so we may identify $\mathcal{X} = G/L$ with an orbit in $\mathcal{F}^+ \times \mathcal{F}^-$. In fact this is the unique open orbit. From the product structure on $\mathcal{F}^+ \times \mathcal{F}^-$, the orbit $\mathcal{X}$ inherits two G–invariant distributions $E^\pm$, where the vector space over a point $x = (x^+, x^-) \in \mathcal{X} \subset \mathcal{F}^+ \times \mathcal{F}^-$ is $E^\pm_x = T_{x^\pm}\mathcal{F}^\pm$. It follows that any $\mathcal{X}$–bundle also comes equipped with such a pair of distributions, which we again call $E^\pm$.

Example 3.1. What exactly are the spaces $G/P^\pm$ and G/L? Let's take a look at these in our favorite setting where $G = \mathrm{PSL}(2,\mathbb{R})$. In this case, the Borel subgroup B is the subgroup of upper triangular matrices. Thus there are no proper subgroups between G and B and we are forced to take $P^+ = B$ and P^- as the subgroup of lower triangular matrices.

Again identifying $I \in G$ with the vertical-pointing tangent vector at $i \in \mathbb{H}^2$, we find that $P^+ \le G$ is exactly the set of vertical-pointing tangent vectors in $T^1(\mathbb{H}^2)$. Indeed, using the notation of Example 2.1, every element of P^+ may be expressed as a product $g_t x_s$, and both of these flows preserve the set of vertical tangent vectors. In particular we see that all elements of P^+ point towards the same boundary point in $\partial\mathbb{H}^2$ (namely ∞). Multiplying this picture by $g \in G$, we find that all vectors in gP^+ point towards the boundary point $g \cdot \infty \in \partial\mathbb{H}^2$. Thus we have a natural identification of G/P^+ with the circle $S^1 = \partial\mathbb{H}^2$. Similarly all vectors in P^- point away from $0 \in \partial\mathbb{H}^2$, and so we again have an identification of G/P^- with $\partial\mathbb{H}^2$. Finally, G/L may be identified with the space of oriented geodesic lines in $\mathbb{H}^2$, which may equivalently be realized as the set

$$\partial(\mathbb{H}^2)^{(2)} = \partial\mathbb{H}^2 \times \partial\mathbb{H}^2 \setminus \{(t,t) \mid t \in \partial\mathbb{H}^2\}.$$

This same basic picture holds for every rank–1 Lie group. That is, whenever G is a rank–1 semisimple Lie group, the only opposite parabolic subgroups are the Borel subgroup B^+ and its opposite B^-, and for both of these there is a natural identification of $G/B^\pm$ with the boundary of the symmetric space G/K. When G has higher rank, there are more possibilities for the opposite parabolic subgroups (P^+, P^-) and the quotients $G/P^\pm$ can be identified with certain *proper* subspaces of the boundary of the symmetric space G/K.

3.1 For Riemannian manifolds

Let N be a closed negatively curved Riemannian manifold, and let $M = T^1(N)$ be its unit tangent bundle. We then have the universal cover $\tilde{N}$ and its unit tangent bundle $\widehat{M} = T^1(\tilde{N})$, which is a $\Gamma = \pi_1(N)$–cover of M. Let ϕ_t denote the geodesic flow on $\widehat{M}$; this descends to the geodesic flow (also denoted ϕ_t) on M.

Now suppose that we have a representation $\rho\colon \Gamma \to G$. The product $\widehat{M} \times \mathcal{X}$ becomes a Γ–space, where Γ acts diagonally on $\widehat{M}$ by deck transformations and on $\mathcal{X}$ via the representation ρ. We denote the quotient under this action by

$$\mathcal{X}_\rho := \widehat{M} \times_\rho \mathcal{X} = \Gamma \backslash (\widehat{M} \times \mathcal{X}).$$

The projection of $\widehat{M} \times \mathcal{X}$ onto the first factor descends to a map $\mathcal{X}_\rho \to M$ giving $\mathcal{X}_\rho$ the structure of an $\mathcal{X}$–bundle over M. Moreover, there is a natural flat connection on the bundle $\widehat{M} \times \mathcal{X} \to \widehat{M}$ (coming from the product structure) and this gives us a flat connection on $\mathcal{X}_\rho$. (Recall that a **connection** on a bundle is simply a choice of a way to lift any smooth path in the base to the total space. **Flat** here just means that the holonomy is locally trivial, that is, each point in the base has a neighborhood on which all closed loops lift to *closed* loops in the total space). Additionally, we emphasize that $\mathcal{X}_\rho$ comes equipped with two vector bundles $E^\pm \to \mathcal{X}_\rho$ (in fact, these are distributions $E^\pm \leq T\mathcal{X}_\rho$) coming the aforementioned distributions $E^\pm$ on the fiber $\mathcal{X}$.

Now here is an important point: The flow ϕ_t on $\widehat{M}$ lifts to a flow on $\widehat{M} \times \mathcal{X}$ defined by $\psi_t(m, x) = (\phi_t m, x)$. Furthermore, this flow is invariant under the action of Γ, and so defines a flow (also called ψ_t) on $\mathcal{X}_\rho$ which lifts ϕ_t. Notice that the product structure gives us a splitting of the tangent bundle

$$T(\widehat{M} \times \mathcal{X}) = T\widehat{M} \oplus T\mathcal{X},$$

and that ψ_t preserves this splitting (meaning that the derivative $D\psi_t$ preserves each subbundle). In addition, $D\psi_t$ acts trivially on $T\mathcal{X}$, so the flow also preserves each of the distributions $E^\pm \leq T\mathcal{X}$. To summarize, the flow ϕ_t on M lifts to a flow ψ_t on $\mathcal{X}_\rho$ that preserves the two distributions $E^\pm$.

Definition 3.2 (Anosov representation)**.** In the situation described above, a representation $\rho\colon \pi_1(N) = \Gamma \to G$ is called (P^+, P^-)–**Anosov** if

1. the bundle $\mathcal{X}_\rho$ admits a section $\sigma\colon M \to \mathcal{X}_\rho$ that is flat along flow lines (i.e., thinking of a flowline as a path $h\colon [0,1] \to M$, the composition of this with the section σ must agree with a lift of h to $\mathcal{X}_\rho$ determined by the flat connection), and
2. the lifted action of ϕ_t (which is determined by the action of ψ_t) on $\sigma^* E^+$ (resp. $\sigma^* E^-$) is expanding (resp. contracting) (see §2.1 for definitions of these terms).

Remark 3.3. In the definition of an Anosov flow (see §2.1), the expanding and contracting properties were measured with respect the Riemannian norm on each tangent bundle. In the setting of Definition 3.2 the bundles $\sigma^* E^\pm$ are not subbundles of the tangent bundle TM, so we don't use the Riemannian metric here. Rather, property (2) above just means that there is some continuous family of norms $(\|\cdot\|_m)_{m \in M}$ on the fibers of the bundles $\sigma^* E^\pm$ so that the expanding/contracting properties are satisfied. Again, since M is compact this does not depend on the choice of the family of norms.

3.2 The equivariant maps

Let us further study the structure of an Anosov representation $\rho\colon \Gamma \to G$, where as above $\Gamma = \pi_1(N)$. A section σ of the bundle $\mathcal{X}_\rho \to M$ is equivalent to a ρ–equivariant $\hat{\sigma}\colon \widehat{M} \to \mathcal{X}$, and it is easy to see that σ is 'flat along flow lines' if and only if $\hat{\sigma}$ is ϕ_t–equivariant. Such a map may instead be considered to be defined on the space

$$\widehat{M}/\phi_t = \partial\tilde{N}^{(2)} = \partial\tilde{N} \times \partial\tilde{N} \setminus \{(t,t) \mid t \in \partial\tilde{N}\}$$

of ordered pairs of distinct boundary points of $\tilde{N}$. Thus the section $\sigma\colon M \to \mathcal{X}_\rho$ in the definition of an Anosov representation is in fact equivalent to a pair of maps

$$\hat{\sigma} = (\xi^+, \xi^-)\colon \partial\tilde{N}^{(2)} \to \mathcal{X} \subset \mathcal{F}^+ \times \mathcal{F}^-.$$

Furthermore, the contracting property of the Anosov representation implies that $\xi^+\colon \partial\tilde{N}^{(2)} \to \mathcal{F}^+$ in fact only depends on the first coordinate, that is, it factors through the projection $\partial\tilde{N}^{(2)} \to \partial\tilde{N}$ onto the first factor. Similarly ξ^- factors through the projection onto the second factor. Thus from an Anosov representation we obtain a pair of ρ–equivariant maps

$$\xi^\pm\colon \partial\tilde{N} \to \mathcal{F}^\pm.$$

These maps are called the **Anosov maps** associated to the Anosov representation. Conversely, given such maps, one may recover the section $\sigma\colon M \to \mathcal{X}_\rho$, and the properties imposed on σ in Definition 3.2 can be reformulated in terms of these Anosov maps.

In summary, the definition of a (P^+, P^-)–Anosov representation $\rho\colon \pi_1(N) \to G$ may be reformulated in terms of the existence of continuous ρ–equivariant maps $\xi^\pm \partial\tilde{N} \to G/P^\pm$ satisfying certain properties which are analogous to those in Definition 3.2 above; see [GW, Proposition 2.7] for details.

3.3 For hyperbolic groups

Whereas Labourie originally formulated the notion of Anosov representations in the setting of negatively curved manifolds (as described in § 3.1 above), Guichard and Wienhard [GW] extended this notion to representations $\rho\colon \Gamma \to G$ of an arbitrary word hyperbolic group Γ.

The key in this definition is to replace the space $(\widehat{M}, \phi_t)$ with an appropriate 'flow space' for the hyperbolic group. Such a space, which we will denote $\widehat{\Gamma}$, was introduced by Gromov [Gro] and further developed by Champetier [Cha], Mineyev [Min] and others. This space comes equipped with an action by the group $\Gamma \times \mathbb{R} \rtimes (\mathbb{Z}/2\mathbb{Z})$, where we should think of Γ as acting by deck transformations, the $\mathbb{R}$ as a 'geodesic flow', and the $\mathbb{Z}/2\mathbb{Z}$ as reversing direction. In particular, every orbit $\Gamma \to \widehat{\Gamma}$ is a quasi-isometry, the $\mathbb{R}$–orbits $\mathbb{R} \to \widehat{\Gamma}$ are quasi-geodesics, and the induced map

$$\widehat{\Gamma}/\mathbb{R} \to \partial\widehat{\Gamma}^{(2)} \cong \partial\Gamma^{(2)} = \partial\Gamma \times \partial\Gamma \setminus \{(t,t) \mid t \in \partial\Gamma\}$$

is a homeomorphism.

While one does not have all the nice manifold machinery in this setting, given a representation $\rho\colon \Gamma \to G$, one can nevertheless consider either ρ–equivariant maps $\xi^\pm\colon \partial\Gamma \to \mathcal{F}^\pm$ or sections of the bundle $\mathcal{X}_\rho = \widehat{\Gamma} \times_\rho \mathcal{X} \to \Gamma\backslash\widehat{\Gamma}$ that are flat along flow lines. In either of these settings, one can then formulate the criterion for being (P^+, P^-)–Anosov in terms of expanding/contracting properties that are directly analogous to those considered in §§3.1–3.2 above. For details, see §2.3 of [GW].

4 L–Cartan projections

Given a hyperbolic group Γ and a representation $\rho\colon \Gamma \to G$, we again form the associated flat bundle $\mathcal{X}_\rho = \widehat{\Gamma} \times_\rho \mathcal{X}$. When attempting to verify whether ρ satisfies the definition of an Anosov representation, it is often difficult to check the expanding/contracting properties because any candidate section $\sigma\colon \Gamma\backslash\widehat{\Gamma} \to \mathcal{X}_\rho$ must be flat, and therefore effectively 'constant', along flowlines. To aid in this task, Guichard and Wienhard introduce the tool of L–Cartan projections:

Recall that up to conjugacy, every pair of opposite parabolic subgroups is determined by choosing some subset of the simple roots in a root system. In particular there are not so many choices for the parabolic subgroups $P^\pm$. Going forward, let's focus on the parabolic subgroups $P^\pm = P^\pm_\Theta$ determined by such a subset Θ of the simple roots Δ.

Recall that $L = P^+ \cap P^-$ is the common Levi component of our parabolic subgroups. We consider its Weyl chamber $\mathfrak{a}_L^+$, which can be viewed as a subalgebra of the Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{g}$. Specifically, $\mathfrak{a}_L^+ = \{a \in \mathfrak{a} \mid \alpha(a) > 0, \text{ for all } \alpha \in \Theta\}$ (recall Θ is a subset of the dual Lie algebra $\mathfrak{a}^*$).

Let M be the maximal compact subgroup of L, and set $\mathcal{Y} = G/M$. Given our representation $\rho\colon \Gamma \to G$, we can then form the flat bundle $\mathcal{Y}_\rho = \widehat{\Gamma} \times_\rho \mathcal{Y}$, which is an L/M–bundle over $\mathcal{X}_\rho$. By lifting a candidate section $\sigma\colon \Gamma\backslash\widehat{\Gamma} \to \mathcal{X}_\rho$ to this larger bundle, one obtains a refined section $\beta\colon \Gamma\backslash\widehat{\Gamma} \to \mathcal{Y}_\rho$ which is *no longer* 'constant' along flowlines. Because the composition of the flowline through $m \in \Gamma\backslash\widehat{\Gamma}$ with β actually moves in the fiber $\mathcal{Y} = G/M$, one may look at the *change* between $\beta(m)$ and $\beta(\phi_t m)$ and measure the Cartan projection of this change. This projection is naturally an element of the Weyl chamber $\mathfrak{a}_L^+$. In this way, one obtains maps

$$\mu_\pm\colon \widehat{\Gamma} \times \mathbb{R} \to \overline{\mathfrak{a}}_L^+$$

which are termed the L–**Cartan projections** of σ. Then for $(m,t) \in (\Gamma\backslash\widehat{\Gamma}) \times \mathbb{R}$, one may define

$$A_\pm(m,t) = \min_{\alpha \in \Delta\backslash\Theta} \alpha(\mu_\pm(\hat{m}, t)).$$

The key here is that these numbers really aren't all that complicated or too difficult to work with. Using these tools, the process of checking for an Anosov representation reduces to checking a few eigenvalues:

Proposition 4.1 (Prop 3.16 of [GW]). *Let $\rho\colon \Gamma \to G$ be a representation, let σ be a section of $\mathcal{X}_\rho$ which is flat along $\mathbb{R}$–orbits, and let $A_\pm$ be as above. The following are equivalent:*

1. *σ is an Anosov section (and so ρ is a (P^+, P^-)–Anosov representation).*
2. *There exist positive constants C, c such that for all $t \geq 0$ and $m \in \Gamma\backslash\widehat{\Gamma}$ one has $A_+(m,t) \geq ct - C$ and $A_-(m,-t) \geq ct - C$.*
3. *There exist positive constants C, c such that for all $t \geq 0$ and $m \in \Gamma\backslash\widehat{\Gamma}$ one has $A_+(m,t) \geq ct - C$.*
4. *$\lim_{t\to\infty} \inf_{m\in\Gamma\backslash\widehat{\Gamma}} A_+(m,t) = \infty$.*

5 Basic properties

One can show that if you are given an Anosov representation $\rho\colon \Gamma \to G$, then the corresponding Anosov maps $\xi^\pm\colon \partial\Gamma \to \mathcal{F}^\pm$ are *unique* (that is, there is just one choice of section $\sigma\colon \Gamma\backslash\widehat{\Gamma} \to \mathcal{X}_\rho$ that will be flat along flow lines and satisfy the expanding/contracting properties).

This uniqueness, together with the density of fixed points of non-torsion $\gamma \in \Gamma$ in $\partial\Gamma$ imply that a representation $\rho\colon \Gamma \to G$ is Anosov if and only if the same holds for the restriction to a finite index subgroup $\Gamma' < \Gamma$, or for the composition $\pi \circ \rho$ with a covering $\pi\colon \widehat{G} \to G$ of Lie groups (where you choose the parabolic subgroups compatibly).

Here are some other basic properties of Anosov representations:

- Every Anosov representation $\rho\colon \Gamma \to G$ is a quasi-isometric embedding; in particular:
  - $\ker(\rho)$ is finite,
  - $\rho(\Gamma) \leq G$ is discrete, and
  - ρ is well-displacing (meaning that γ and $\rho(\gamma)$ have uniformly comparable translation lengths)
- For a fixed parabolic subgroup $P < G$, the set of P–Anosov representations in $\mathrm{Hom}(\Gamma, G)$ is open.
- When G has rank one, there is exactly one conjugacy class of parabolic subgroups $P < G$ (so we can unambiguously talk about a representation begin Anosov), and the following are equivalent for a representation $\rho\colon \Gamma \to G$:
  - ρ is Anosov
  - there exits a continuous, equivariant, and injective map $\xi\colon \partial\Gamma \to G/P$
  - ρ is a quasi-isometry
  - $\ker(\rho)$ is finite and $\rho(\Gamma)$ is convex convex cocompact (i.e., acts cocompactly on some convex subset of the symmetric space G/K).

Finally, various well-studied types of representations are Anosov, such as maximal representations and representations in the Hitchin component.

References

[Cha] Christophe Champetier. Petite simplification dans les groupes hyperboliques. *Ann. Fac. Sci. Toulouse Math. (6)*, 3(2):161–221, 1994.

[Col] Brian Collier. Semi-simple lie groups, 2013. Summary of talk at the *Workshop on higher Teichmüller-Thurston theory* in Northport, ME.

[Gro] M. Gromov. Hyperbolic groups. In *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263. Springer, New York, 1987.

[GW] Olivier Guichard and Anna Wienhard. Anosov representations: domains of discontinuity and applications. *Invent. Math.*, 190(2):357–438, 2012.

[Lab] François Labourie. Anosov flows, surface groups and curves in projective space. *Invent. Math.*, 165(1):51–114, 2006.

[Min] Igor Mineyev. Flows and joins of metric spaces. *Geom. Topol.*, 9:403–482, 2005.

[Pla] Joseph F. Plante. Anosov flows. *Amer. J. Math.*, 94:729–754, 1972.