GEOMETRIC CHARACTERIZATION OF HITCHIN REPRESENTATIONS

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ABSTRACT. These notes are an attempt of summary of Labourie-Guichard's characterization of Hitchin representations established in [La] and [Gui_3] via the notion of hyperconvexity.

1. HYPERCONVEXITY

We discuss the notions of hyperconvex representation and Anosov representations. In particular, we prove that hyperconvex representations are Anosov.

1.1. Hyperconvex representations.

Definition. A flag curve $\mathcal{F}\colon \partial_\infty\widetilde{S} \to \mathrm{Flag}(\mathbb{R}^n)$ is *hyperconvex* if it satisfies the following two conditions:

(1) for every k–tuple of distinct points $x_1, \ldots, x_k \in \partial_\infty\widetilde{S}$, for every k–tuple of integers $m_1, \ldots, m_k$ such that $m_1 + \cdots + m_k = n$,
$$\mathbb{R}^n = \mathcal{F}^{(m_1)}(x_1) \oplus \cdots \oplus \mathcal{F}^{(m_k)}(x_k);$$

(2) for every k–tuple of distinct points $x_1, \ldots, x_k \in \partial_\infty\widetilde{S}$, for every k–tuple of integers $m_1, \ldots, m_k$ such that $m_1 + \cdots + m_k = m \leq n$,
$$\lim_{\substack{(x_i)\to x \\ x_i\neq x}} \mathcal{F}^{(m_1)}(x_1) \oplus \cdots \oplus \mathcal{F}^{(m_k)}(x_k) = \mathcal{F}^{(m)}(x).$$

Note that a hyperconvex flag curve is in particular continuous. Besides, the image of the projective curve $\mathcal{F}^{(1)}\colon \partial_\infty\widetilde{S} \to \mathbb{RP}^{n-1}$ is a $\mathcal{C}^1$–embedded curve.

Definition. A homomorphism $\rho\colon \pi_1(S) \to \mathrm{PSL}_n(\mathbb{R})$ is a *hyperconvex representation* if there exists a ρ–equivariant, hyperconvex flag curve $\mathcal{F}_\rho\colon \partial_\infty\widetilde{S} \to \mathrm{Flag}(\mathbb{R}^n)$.

Example. Let $\rho_0\colon \pi_1(S) \to \mathrm{PSL}_n(\mathbb{R})$ be a *n–Fuchsian representation*, namely ρ_0 is a homomorphism of the form
$$\rho_0 = \iota \circ r$$
where: $r\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{R})$ is a Fuchsian homomorphism; and $\iota\colon \mathrm{PSL}_2(\mathbb{R}) \to \mathrm{PSL}_n(\mathbb{R})$ is the preferred homomorphism defined by the n–dimensional, irreducible representation of $\mathrm{SL}_2(\mathbb{R})$ into $\mathrm{SL}_n(\mathbb{R})$. Then ρ_0 is hyperconvex. Identify $\partial_\infty\widetilde{S}$ with $\mathbb{RP}^1$ viewed as the set of nonzero homogenous polynomials of the form $aX + bY$ up to scalar multiplication. The associated equivariant flag curve $\mathcal{F}_{\rho_0}$ is defined as follows: for every $i = 1, \ldots, n-1$, $\mathcal{F}^{(i)}_{\rho_0}\big([aX+bY]\big)$ is the i–dimensional subspace of homogenous polynomials of the form $a_0X^{n-1} + a_1X^{n-2}Y + \cdots + a_{n-1}Y^{n-1}$ that are multiples of $(aX+bY)^{n-i}$. One easily verifies that $\mathcal{F}_{\rho_0}$ is hyperconvex as it satisfies

both conditions (1) and (2). The projective curve $\mathcal{F}_{\rho_0}^{(1)}$ is called the *Veronese's embedding of* $\mathbb{RP}^1$ *in* $\mathbb{RP}^{n-1}$.

Exercise. Show that the "Veronese" flag curve is hyperconvex.

1.2. **Anosov representations.** Set $\bar{\mathbb{R}}^n = \mathbb{R}^n/\{\pm \mathrm{Id}\}$; note that $\mathrm{PSL}_n(\mathbb{R})$ acts on $\bar{\mathbb{R}}^n$. Given a homomorphism $\rho\colon \pi_1(S) \to \mathrm{PSL}_n(\mathbb{R})$, let $T^1S \times_\rho \bar{\mathbb{R}}^n \to T^1S$ be the flat twisted $\bar{\mathbb{R}}$–bundle associated with ρ.

Definition. A homomorphism $\rho\colon \pi_1(S) \to \mathrm{PSL}_n(\mathbb{R})$ is an *Anosov representation* if there exists a ρ–equivariant flag curve $\mathcal{F}_\rho\colon \partial_\infty \widetilde{S} \to \mathrm{Flag}(\mathbb{R}^n)$ that satisfies the following two conditions:

(1) for every pair of distinct points $x, y \in \partial_\infty \widetilde{S}$, for every $k = 1, \dots, n-1$,
$$\mathbb{R}^n = \mathcal{F}_\rho^{(k)}(x) \oplus \mathcal{F}_\rho^{(n-k)}(y);$$

(2) let $(G_t)_{t\in\mathbb{R}}$ be the lift in the flat bundle $T^1S \times_\rho \bar{\mathbb{R}}^n$ of the geodesic flow $(g_t)_{t\in\mathbb{R}}$; the flag curve $\mathcal{F}_\rho$ provides a line splitting $V_1 \oplus \cdots \oplus V_n$ of $T^1S \times_\rho \mathbb{R}^n \to T^1S$ that is invariant under the action of $(G_t)_{t\in\mathbb{R}}$; we require the action of $(G_t)_{t\in\mathbb{R}}$ to be *Anosov* in the following sense; there exist a Riemannian metric $\| \ \|$ on the fibres of $T^1S \times_\rho \bar{\mathbb{R}}^n$ and some constants $A, a > 0$ such that, for every $i = 1, \dots, n-1$, for every $t > 0$, for every $u \in T^1S$, for every unit vectors $X_i(u) \in V_i(u)$ and $X_{i+1}(u) \in V_{i+1}(u)$,
$$\frac{\|G_t X_i(u)\|_{g_t(u)}}{\|G_t X_{i+1}(u)\|_{g_t(u)}} \leq A e^{-at}.$$

A consequence of the Anosov dynamics is that an Anosov representation ρ admits a unique equivariant flag curve $\mathcal{F}_\rho$ satisfying (1) and (2). In addition, $\mathcal{F}_\rho\colon \partial_\infty \widetilde{S} \to \mathrm{Flag}(\mathbb{R}^n)$ is Hölder continuous. See [La, Gui$_3$, GuiW] for details.

Example. A n–Fuchsian $\rho_0 = \iota \circ r$ is Anosov. It is convenient to look at the situation in the universal cover $T^1\widetilde{S} \times \bar{\mathbb{R}}^n$. Given a base point $\widetilde{u}_0 \in T^1\widetilde{S}$, we can identify $T^1\widetilde{S}$ with $\mathrm{PSL}_2(\mathbb{R})$; the action of the geodesic flow $(g_t)_{t\in\mathbb{R}}$ on T^1S then identifies with the right action of the subgroup $\left\{\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2}\end{pmatrix}\right\}_{t\in\mathbb{R}}$ on $\mathrm{PSL}_2(\mathbb{R})$.

The Veronese flag curve $\mathcal{F}_{\rho_0}$ provides a line splitting $\widetilde{V}_1 \oplus \cdots \oplus \widetilde{V}_n$ of $T^1\widetilde{S} \times \bar{\mathbb{R}}^n$ that is invariant under the action of the lift $(G_t)_{t\in\mathbb{R}}$. Now, since T^1S is compact, to show that the flow $(G_t)_{t\in\mathbb{R}}$ is Anosov in the sense of (2), we may choose any suitable metric on the bundle $T^1S \times_{\rho_0} \bar{\mathbb{R}}^n$ for which the action is Anosov.

Pick a metric $\| \ \|_{\widetilde{u}_0}$ on the fibre $\bar{\mathbb{R}}^n_{\widetilde{u}_0}$ above $\widetilde{u}_0$; for every $\widetilde{u} = g\widetilde{u}_0 \in T^1\widetilde{S}$ where $g \in \mathrm{PSL}_2(\mathbb{R})$, for every $X \in \bar{\mathbb{R}}^n_{\widetilde{u}}$, set
$$\|X\|_{\widetilde{u}} := \left\| \iota(g)^{-1} X \right\|_{\widetilde{u}_0}.$$

This defines a metric on the fibres of the bundle $T^1\widetilde{S} \times \bar{\mathbb{R}}^n$. By construction, it is invariant under the left action of $\mathrm{PSL}_2(\mathbb{R})$ and thus descends to a well-defined metric on the bundle $T^1S \times_{\rho_0} \bar{\mathbb{R}}^n$. Observe that, if $\widetilde{u} = g\widetilde{u}_0$ where $g \in \mathrm{PSL}_2(\mathbb{R})$, then $g_t(\widetilde{u}) = (ga_tg^{-1})\widetilde{u}$ where $a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$. Because of the flat connection,

$$\begin{aligned} \|G_tX\|_{g_t(\widetilde{u})} &= \|X\|_{g_t(\widetilde{u})} \\ &= \|X\|_{(ga_tg^{-1})\widetilde{u}} \\ &= \left\|\iota(ga_tg^{-1})^{-1}X\right\|_{\widetilde{u}_0} \\ &= \left\|\iota(g)\iota(a_t)^{-1}\iota(g)^{-1})X\right\|_{\widetilde{u}_0}. \end{aligned}$$

It is easy to conclude from the above relation and the definition of the homomorphism $\iota\colon \mathrm{PSL}_2(\mathbb{R}) \to \mathrm{PSL}_n(\mathbb{R})$ that the action of the flow $(G_t)_{t\in\mathbb{R}}$ on the line splitting $\widetilde{V}_1 \oplus \cdots \oplus \widetilde{V}_n \to T^1S$ provided by the Veronese flag curve $\mathcal{F}_{\rho_0}$ satisfies the Anosov condition (2).

1.3. **Hyperconvex implies Anosov.** We now show the following result.

Theorem 1. *Let ρ be a hyperconvex representation. Then ρ is Anosov.*

Sketch of the proof. Consider the line bundle $V_{i+1}^* \otimes V_i \to T^1S$, endowed with the following metric $\| \ \|$ on the fibres: for every $u \in T^1S$, for every $\psi \in V_{i+1}^* \otimes V_i(u)$, set

$$\|\psi\|_u = \left| \frac{\langle \alpha_i, \psi(X_{i+1})\rangle \langle \alpha_{i+1}, Z_0\rangle}{\langle \alpha_{i+1}, X_{i+1}\rangle \langle \alpha_i, Z_0\rangle} \right|$$

where: $\widetilde{u} = (x_+, x_0, x_-) \in T^1\widetilde{S}$ is a lift of u; $Z_0 \in \mathcal{F}_\rho^{(1)}(x_0)$; $X_{i+1} \in \widetilde{V}_{i+1}(\widetilde{u})$; and α_j is a linear form with $\ker(\alpha_j) = \mathcal{F}_\rho^{(j-1)}(x_+) \oplus \mathcal{F}_\rho^{(n-j)}(x_-)$. Note that, because of the transversality condition (1) of a hyperconvex flag curve, $\| \ \|$ is well defined; moreover, it is clearly independent of the choices of Z_0, X_{i+1} and α_j.

To prove the condition (2), it is enough to show that the action of the flow $(G_t)_{t\in\mathbb{R}}$ on the line bundle $V_{i+1}^* \otimes V_i$ is exponentially contracting. To do so, we begin with proving that $\|G_t\psi\|_{g_t(u)}$ converges to 0 as t goes to $+\infty$.

Because of the flat connection,

$$\frac{\|G_t\psi\|_{g_t(u)}}{\|\psi\|_u} = \frac{\|\psi\|_{g_t(u)}}{\|\psi\|_u} = \left| \frac{\langle \alpha_{i+1}, Z_t\rangle \langle \alpha_i, Z_0\rangle}{\langle \alpha_i, Z_t\rangle \langle \alpha_{i+1}, Z_0\rangle} \right|$$

where: $g_t(\widetilde{u}) = (x_+, x_t, x_-)$, and $Z_t \in \mathcal{F}_\rho^{(1)}(x_t)$. Since $\lim_{t\to+\infty} \mathcal{F}_\rho^{(1)}(x_t) \oplus \mathcal{F}_\rho^{(i-1)}(x_+) = \mathcal{F}_\rho^{(i)}(x_+)$, we may assume that Z_t converges to a vector $Z_\infty \in \mathcal{F}_\rho^{(i)}(x_+) - \mathcal{F}_\rho^{(i-1)}(x_+)$. As a result,

$$\begin{aligned} &\lim_{t\to+\infty} \langle \alpha_i, Z_t\rangle = \langle \alpha_i, Z_\infty\rangle \neq 0; \\ &\lim_{t\to+\infty} \langle \alpha_{i+1}, Z_t\rangle = \langle \alpha_{i+1}, Z_\infty\rangle = 0. \end{aligned}$$

It follows that $\lim_{t\to+\infty} \|G_t\psi\|_{g_t(u)} = 0$.

The exponential contraction comes as a consequence of the compacity of T^1S. First of all, note that, since we are dealing with a flow, it is enough to show that there exists some $t_0 > 0$ so that, for every $t \geq t_0$, for every $u \in T^1S$, for every $\psi \in V_{i+1}^* \otimes V_i(u)$,

$$\|G_t\psi\|_{g_t(u)} < \frac{1}{2}\|\psi\|_u.$$

By contradiction, let $t_q \to +\infty$, $\widetilde{u}_q = (x_{+,q}, x_q, x_{-,q}) \in T^1\widetilde{S}$ and $\psi_q \in V_{i+1}^* \otimes V_i(u_q)$ be sequences for which $\left\|G_{t_q}\psi_q\right\|_{g_{t_q}(u_q)} \geq 1/2\,\|\psi_q\|_{g_{t_q}(u_q)}$. By compacity of

T^1S and $\pi_1(S)$–invariance, we may assume that $\widetilde{u}_q$ converges to $\widetilde{u}_\infty = (x_+, x_\infty, x_-)$. Thus: set $g_{t_q}(\widetilde{u}_q) = (x'_{+,q}, x'_q, x'_{-,q})$; we have $x'_q \longrightarrow_{q\to+\infty} x_+$. Let

$$\frac{\left\|G_{t_q}\psi_q\right\|_{g_{t_q}(u_q)}}{\left\|\psi_q\right\|_{u_q}} = \frac{\left\|\psi_q\right\|_{g_{t_q}(u_q)}}{\left\|\psi_q\right\|_{u_q}} = \left|\frac{\left\langle \alpha_{i+1}, Z'_q\right\rangle \left\langle \alpha_i, Z_q\right\rangle}{\left\langle \alpha_i, Z'_q\right\rangle \left\langle \alpha_{i+1}, Z_q\right\rangle}\right| > \frac{1}{2}$$

where: $Z_q \in \mathcal{F}_\rho^{(1)}(x_q)$; and $Z'_q \in \mathcal{F}_\rho^{(1)}(x'_q)$. Since $\lim_{q\to+\infty} \mathcal{F}_\rho^{(1)}(x'_q) \oplus \mathcal{F}_\rho^{(i-1)}(x_+) = \mathcal{F}_\rho^{(i)}(x_+)$, we may assume that Z'_q converges to a vector $Z'_\infty \in \mathcal{F}_\rho^{(i)}(x_+) - \mathcal{F}_\rho^{(i-1)}(x_+)$, which, by the previous reasoning, yields a contradiction. See [Gui$_3$, §3], Proposition 18 for additional details. □

Note that in the above proof, we only make use of the transversality condition for a triple of flags, and of the limit condition (2) in the case where $i = 1$. Therefore, the hyperconvex condition appears to be notably more restrictive than Anosov.

2. Openess of hyperconvex representations

2.1. Anosov 3–hyperconvex representations.

Definition. A flag curve $\mathcal{F}\colon \partial_\infty\widetilde{S} \to \mathrm{Flag}(\mathbb{R}^n)$ is 3–*hyperconvex* if for every triple of distinct points x, y, $z \in \partial_\infty\widetilde{S}$, for every triple of integers j, k, l,

$$\mathbb{R}^n = \mathcal{F}^{(j)}(x) \oplus \mathcal{F}^{(k)}(y) \oplus \mathcal{F}^{(l)}(z).$$

Let $\mathcal{A}^3(S) \subset \mathcal{R}_{\mathrm{PSL}_n(\mathbb{R})}(S)$ be the set of Anosov representations that are 3–hyperconvex. Note that $\mathcal{A}^3(S)$ contains the set of hyperconvex representations $\mathcal{H}(S)$.

Lemma 2. *The set $\mathcal{A}^3(S)$ is open in $\mathcal{R}_{\mathrm{PSL}_n(\mathbb{R})}(S)$.*

Proof. Let ρ_0 be an Anosov representation. There exists an open subset $U \ni \rho_0$ such that the map

$$\begin{aligned}\Phi\colon U \times \partial_\infty\widetilde{S} &\to \mathrm{Flag}(\mathbb{R}^n)\\ (\rho, x) &\mapsto \mathcal{F}_\rho(x)\end{aligned}$$

is continuous.

Let $\partial_\infty\widetilde{S}^{3,*}$ be the set of triple of distinct points in $\partial_\infty\widetilde{S}$; recall that $\partial_\infty\widetilde{S}^{3,*}$ identifies with $T^1\widetilde{S}$. Consider the map

$$\begin{aligned}\Psi\colon U' \times \partial_\infty\widetilde{S}^{3,*} &\to \mathbb{N}\\ \big(\rho, (x,y,z)\big) &\mapsto \dim\big(\mathcal{F}_\rho(x) + \mathcal{F}_\rho(y) + \mathcal{F}_\rho(z)\big).\end{aligned}$$

By compacity of T^1S and $\pi_1(S)$–invariance of Ψ, it follows from the continuity of Φ that there exists $U' \subset U$ for which Ψ is constant and equal to n. See [Gui$_3$, §4], Proposition 20. □

The flag curve of an Anosov 3–hyperconvex representation satisfies the following regularity property.

Proposition 3. *Let $\mathcal{F}_\rho\colon \partial_\infty\widetilde{S} \to \mathrm{Flag}(\mathbb{R}^n)$ be the flag curve of an Anosov 3–hyperconvex representation ρ. Then, for every integers $m = k + l \le n$, for every $x \in \partial_\infty\widetilde{S}$,*

$$\lim_{\substack{(y,z)\to x\\ y\neq z}} \mathcal{F}_\rho^{(k)}(y) \oplus \mathcal{F}_\rho^{(l)}(z) = \mathcal{F}_\rho^{(m)}(x).$$

Note that the above limit implies the existence of a tangent line to the projective curve $\mathcal{F}_\rho^{(1)}$: the (image of the) curve $\mathcal{F}_\rho$ is $\mathcal{C}^1$. Besides, Theorem 1 shows that, if $\mathcal{F}_\rho$ is "smooth" enough, then ρ is Anosov. As it will become more apparent in the proof, Proposition 3 to some extent shows that the Anosov property is necessary to guarantee enough regularity for the flag curve $\mathcal{F}_\rho$.

Sketch of the proof in the case where $l = 1$. For $k \leq n$, consider the map

$$\eta^k\colon (y,z) \mapsto \begin{cases} \mathcal{F}_\rho^{(k-1)}(y) \oplus \mathcal{F}_\rho^{(1)}(z) & \text{if } y \neq z; \\ \mathcal{F}_\rho^{(k)}(y) & \text{if } y = z. \end{cases}$$

We want to show that η^k is continuous; clearly, we are only concerned with points of the form (y,y). Let $I \subset \partial_\infty \widetilde{S}$ be an interval; and let x_- be a point in the complement $\partial_\infty \widetilde{S} - I$. For every $y, z \in I$, set $\xi^{n-1}(y,z) = \eta^k(y,z) \oplus \mathcal{F}_\rho^{(n-k-1)}(x_-)$.

For every $x_+ \in I$, it follows (see exercise below) from the Anosov property for the line decomposition $V_1 \oplus \cdots \oplus V_n \to T^1 S$ and the 3–hyperconvexity that

$$\lim_{x \to x_+} \xi^{n-1}(x_+, x) = \xi^{n-1}(x_+, x_+).$$

Moreover, pick $x_0 \in I$. The contraction property enables us to show that the convergence is uniform, namely, for every $\varepsilon > 0$, for every $y^+ \in I$, there exist an interval $I' \subset I$ that contains y^+, and a constant $\delta > 0$, such that, for every $x_+, x \in I'$ with $\mathrm{dist}_{\partial_\infty \widetilde{S}}(x_+, x) \leq \delta$,

$$\mathrm{dist}_{\mathrm{Gr}^{n-1}(\mathbb{R}^n)}\big(\xi^{n-1}(x_+, x), \xi^{n-1}(x_+, x_+)\big) < \varepsilon.$$

Hence, by continuity of the map $y \mapsto \xi^{n-1}(y,y)$,

$$\lim_{(x_+, x) \to y_+} \xi^{n-1}(x_+, x) = \xi^{n-1}(y_+, y_+).$$

Finally, since $\eta^k = \xi^{n-1} \cap \eta^{k+1}$, we obtain the continuity of the map η^k for all $k \leq n-1$ by descending induction. The case when $l \geq 2$ is identical. See [Gui$_3$, §4], Proposition 20 for additional details. ☐

Exercise. Let $x_+ \in I$, and let $x_q \in I$ be a sequence converging to x_+ as $q \to \infty$. Using the techniques introduced in the proof of Theorem 1, show that, for every $\varepsilon > 0$, there exist an integer Q and a constant $C > 0$ such that, for every sequence of vectors $Z_q \in \mathcal{F}_\rho^{(k-1)}(x_+) \oplus \mathcal{F}_\rho^{(1)}(x_q) \oplus \mathcal{F}_\rho^{(n-k-1)}(x_-)$,

$$\mathrm{dist}_{\mathbb{R}^n}\big(Z_q, \mathcal{F}_\rho^{(k)}(x_+) \oplus \mathcal{F}_\rho^{(n-k-1)}(x_-)\big) \leq C\,\|Z_q\|\,\varepsilon.$$

Conclude that $\lim_{x \to x_+} \xi^{n-1}(x_+, x) = \xi^{n-1}(x_+, x_+)$.

2.2. **Openess.** Let $\mathcal{H}(S)$ be the set of hyperconvex representations.

Theorem 4. *$\mathcal{H}(S)$ is open in $\mathcal{R}_{\mathrm{PSL}_n(\mathbb{R})}(S)$.*

Let $\mathrm{Gr}^{i,+}(\mathbb{R}^n)$ be the Grassmannians of oriented vector spaces $\mathrm{Gr}^{i,+}(\mathbb{R}^n)$; it a 2–cover of $\mathrm{Gr}^i(\mathbb{R}^n)$. The observation below will come in very handy.

Lemma 5. *Let $\mathcal{F}\colon I \to \mathrm{Flag}(\mathbb{R}^n)$ be a hyperconvex flag curve defined on an (oriented) interval $I \subset \partial_\infty \widetilde{S}$. For every $i = 1, \ldots, n-1$, suppose that the map $\mathcal{F}^{(i)}\colon I \to \mathrm{Gr}^i(\mathbb{R}^n)$ lifts to $\mathcal{F}^{(i),+}\colon I \to \mathrm{Gr}^{i,+}(\mathbb{R}^n)$ so that*

$$\lim_{(y > z) \to x} \mathcal{F}^{(i-1),+}(y) \oplus \mathcal{F}^{(1),+}(z) = \mathcal{F}^{(i),+}(x).$$

Then there exists an orientation $\mathbb{R}^{n,+}$ *of* $\mathbb{R}^n$ *such that, for every* $n_1+\cdots+n_k=n$, *for every* $x_1>\cdots>x_k$,

$$\mathbb{R}^{n,+}=\mathcal{F}^{(n_1),+}(x_1)\oplus\cdots\oplus\mathcal{F}^{(n_k),+}(x_k).$$

Proof. Left to the reader as an exercise. □

Sketch of the proof of Theorem 4. The proof requires several steps. The key idea is the following: if $U\subset\mathcal{A}^3(S)$ is a contractible neighborhood of a hyperconvex representation ρ_0, then all representations in U are hyperconvex.

Let $\rho\in U$. By induction on k, we prove that, for every $x\in\partial_\infty\widetilde{S}$,

$$\mathrm{H}(k)\colon\begin{cases}\text{for every } k\text{–tuples of integers } m_1,\ \dots\ ,\ m_k,\\[2ex] \lim_{\substack{(x_i)\to x\\ x_i\neq x}}\mathcal{F}_\rho^{(m_1)}(x_1)\oplus\cdots\oplus\mathcal{F}_\rho^{(m_k)}(x_k)=\mathcal{F}_\rho^{(m)}(x).\end{cases}$$

where $m=m_1+\cdots+m_k\le n-1$.

Note that H(1) is true by continuity of the flag map $\mathcal{F}_\rho$, and H(2) is true by Proposition 3.

Let us focus attention on the special case of H(3) where $m_1=n-3$, $m_2=1$, $m_3=1$. We want to show that

$$\lim_{\substack{(x_i)\to x\\ x_i\neq x}}\mathcal{F}_\rho^{(n-3)}(x_1)\oplus\mathcal{F}_\rho^{(1)}(x_2)\oplus\mathcal{F}_\rho^{(1)}(x_3)=\mathcal{F}_\rho^{(n-1)}(x).\tag{1}$$

Let $I\subset\partial_\infty\widetilde{S}$ be an interval that contains x. Pick an orientation on I. It is enough to analyze the case where the above limit is taken under the extra condition that $x_1>x_2>x_3$. The advantage then resides in the possibility to lift the situation in the Grassmannians of oriented vector spaces $\mathrm{Gr}^{i,+}(\mathbb{R}^n)$. More precisely, let $\mathcal{F}_\rho^{(1),+}\colon I\to\mathrm{Gr}^{1,+}(\mathbb{R}^n)$ that lifts $\mathcal{F}_\rho^{(1)}$; for every $k\le n-1$, define a lift $\mathcal{F}_\rho^{(k),+}$ and an orientation $\mathbb{R}^{n,+}$ for $\mathbb{R}^n$ via the following relations:

$$\mathcal{F}_\rho^{(k+1),+}(x)=\lim_{(y>z)\to x}\mathcal{F}_\rho^{(k),+}(y)\oplus\mathcal{F}_\rho^{(1),+}(z);$$

$$\mathbb{R}^{n,+}=\mathcal{F}_\rho^{(n-1),+}(y)\oplus\mathcal{F}_\rho^{(1),+}(z)\text{ if } y>z.$$

We begin with studying cluster points of converging sequences. Let $(x>)\ x_{1,q}>x_{2,q}>x_{3,q}$ be a triple of sequences that converge to $x\in\partial_\infty\widetilde{S}$. Assume that the sequence

$$P_q^{(n-1),+}=\mathcal{F}_\rho^{(n-3),+}(x_{1,q})\oplus\mathcal{F}_\rho^{(1),+}(x_{2,q})\oplus\mathcal{F}_\rho^{(1),+}(x_{3,q})$$

converges to the oriented hyperplan $P^{(n-1),+}\in\mathrm{Gr}^{n-1,+}(\mathbb{R}^n)$. By Proposition 3, $P^{(n-1)}$ contains $\mathcal{F}_\rho^{(n-2)}(x)$. Hence, for $w<x$, the sequence of oriented lines (3–hyperconvexity!)

$$Z_q^{(1),+}=P_q^{(n-1),+}\cap\mathcal{F}_\rho^{(2),+}(w)$$

converges to the oriented line $Z^{(1),+}=P^{(n-1),+}\cap\mathcal{F}_\rho^{(2),+}(w)$. We want to prove that

$$Z^{(1),+}=\mathcal{F}_\rho^{(n-1),+}(x)\cap\mathcal{F}_\rho^{(2),+}(w)$$

which will imply that $P^{(n-1),+}=\mathcal{F}_\rho^{(n-1),+}(x)$.

Let $D_q^{(n-1),+}$ and $E_q^{(n-1),+}$ be the sequences of oriented hyperplans defined by

$$\begin{aligned} D_q^{(n-1),+} &= \mathcal{F}_\rho^{(n-2),+}(x_{1,q}) \oplus \mathcal{F}_\rho^{(1),+}(x_{3,q}) \\ E_q^{(n-1),+} &= \mathcal{F}_\rho^{(n-3),+}(x_{1,q}) \oplus \mathcal{F}_\rho^{(2),+}(x_{3,q}). \end{aligned}$$

A first step is to show that both sequences converge to the oriented hyperplan $\mathcal{F}^{(n-1),+}(x)$. The convergence without the orientation nonsense is guaranteed by Proposition 3. Now, $\lim_{q\to+\infty} D_q^{(n-1),+} = \mathcal{F}^{(n-1),+}(x)$ is true by definition of the lift $\mathcal{F}^{(n-1),+}$. For the sequence $E_q^{(n-1),+}$, it is enough to show that

$$E_q^{(n-1),+} \oplus \mathcal{F}_\rho^{(1),+}(w) = \mathcal{F}_\rho^{(n-1),+}(x) \oplus \mathcal{F}_\rho^{(1),+}(w) (= \mathbb{R}^{n,+}).$$

This is the moment when the orientation nonsense plays a central rôle. The open subset U being contractile, we can deform the flag curve $\mathcal{F}_\rho$ into the hyperconvex flag curve $\mathcal{F}_{\rho_0}$ through a continuous family of Anosov 3–hyperconvex curves $(\mathcal{F}_{\rho_t})_{t\in[0,1]} \subset U$ with $\mathcal{F}_{\rho_1} := \mathcal{F}_\rho$. Observe then that the lift $\mathcal{F}_{\rho,+}$ for $\mathcal{F}_\rho$ extends a lift $(\mathcal{F}_{\rho_t,+})_{t\in[0,1]}$ so that, for every $t \in [0,1]$,

$$\mathcal{F}_{\rho_t}^{(n-3),+}(x_{1,q}) \oplus \mathcal{F}_{\rho_t}^{(2),+}(x_{3,q}) \oplus \mathcal{F}_{\rho_t}^{(1),+}(w)$$

defines the same orientation on $\mathbb{R}^n$. When $t = 0$, the Lemma 5 applies to the hyperconvex flag curve $\mathcal{F}_{\rho_0}$: it guarantees that the orientations "behave well" when we take limits. As a result,

$$\begin{aligned} E_q^{(n-1),+} \oplus \mathcal{F}_\rho^{(1),+}(w) &= \mathcal{F}_{\rho_1}^{(n-3),+}(x_{1,q}) \oplus \mathcal{F}_{\rho_1}^{(2),+}(x_{3,q}) \oplus \mathcal{F}_{\rho_1}^{(1),+}(w) \\ &= \mathcal{F}_{\rho_0}^{(n-3),+}(x_{1,q}) \oplus \mathcal{F}_{\rho_0}^{(2),+}(x_{3,q}) \oplus \mathcal{F}_{\rho_0}^{(1),+}(w) \\ &= \mathcal{F}_{\rho_0}^{(n-1),+}(x) \oplus \mathcal{F}_{\rho_0}^{(1),+}(w) \\ &= \mathcal{F}_{\rho_1}^{(n-1),+}(x) \oplus \mathcal{F}_{\rho_1}^{(1),+}(w). \end{aligned}$$

We can conclude via the following observation. Consider the two sequences

$$\begin{aligned} &Z^{(1),+} \oplus D_q^{(n-1),+} \\ &Z^{(1),+} \oplus E_q^{(n-1),+} \end{aligned}$$

of $\mathrm{Gr}^{n,+}(\mathbb{R}^n)$. It is easy to see that these sequences have opposite orientations. Moreover, recall that the limits of both sequences each contain $Z^{(1),+}$ and $\mathcal{F}_\rho^{(n-1),+}(x)$. Therefore, $Z^{(1)}$ must be contained in $\mathcal{F}_\rho^{(n-1)}(x)$.

Regarding the existence of the limit (1) (un peu passé sous silence..): it comes as a consequence that (nonoriented and oriented) Grassmannians are compact spaces. Therefore, every sequence that admits a unique cluster point is convergent. See [Gui$_3$, §5], Theorem 23 for additional details. □

3. Closedness of hyperconvex representations

Theorem 6. *The set of hyperconvex representations $\mathcal{H}(S)$ is equal to the union of the Hitchin components of $\mathcal{R}_{\mathrm{PSL}_n(\mathbb{R})}(S)$.*

The proof of Theorem 6 strongly relies on algebraic group techniques; in other words, on aime ou on n'aime pas... It makes great use of results established in [Gui$_2$].

Here are two results that are essential in the following; the first one is a rather easy observation, while the second one requires hard work and is one of the main keys in the proof of Theorem 6.

Lemma 7. *Let ρ be a hyperconvex representation. Then ρ is strongly irreducible.*

Proof. Left to the reader as an exercise. See [Gui_3, §3], Propositions 14 and 15. □

Lemma 8. *Let ρ be a representation that is strongly irreducible and limit of hyperconvex representations. Then ρ is hyperconvex.*

Proof. Left to the reader as a nightmare. See [La, §9]. □

3.1. **The case where the genus is "large".** Let us denote by $\overline{\mathcal{H}(S)}$ be the adherence of the set of hyperconvex representations $\mathcal{H}(S)$. If the genus of the surface is large enough (in relation to n), Theorem 6 comes as a consequence of the following proposition, that, along with Lemma 8, constitute the keys of the proof.

Proposition 9. *$\overline{\mathcal{H}(S)}$ is open in $\mathcal{R}_{\mathrm{PSL}_n(\mathbb{R})}(S)$.*

Proof. See [Gui_3, §6], Proposition 26. □

As a consequence of Proposition 9, since $\mathcal{H}(S)$ contains the n–Fuchsian representations, $\overline{\mathcal{H}(S)}$ contains the Hitchin components.

Sketch of the proof of Theorem 6. By Proposition 9, $\overline{\mathcal{H}(S)}$ is the union of connected components of $\mathcal{R}_{\mathrm{PSL}_n(\mathbb{R})}(S)$; in particular, since $\mathcal{H}(S)$ contains all n–Fuchsian representations, $\overline{\mathcal{H}(S)}$ contains the Hitchin components. Besides, it is a consequence of [Hit] that all representations in the Hitchin components are strongly irreducible; hence, by Lemma 8,

$$\overline{\mathcal{H}(S)} \cap \text{Hitchin components} = \mathcal{H}(S) \cap \text{Hitchin components}$$

Finally, again due to [Hit], representations that are not Hitchin can be deformed into representations valued in the compact Lie group $\mathrm{PSO}_n(\mathbb{R})$; as a result, such components can not contain any discrete representation. □

3.2. **The general case.** The key lies in two simple observations that enable us to go from the large genus case back to the small genus case.

Lemma 10. *Let Γ be a finite index subgroup of $\pi_1(S)$. Then*

(1) *$\rho\colon \pi_1(S) \to \mathrm{PSL}_n(\mathbb{R})$ is hyperconvex if and only if $\rho\colon \Gamma \to \mathrm{PSL}_n(\mathbb{R})$ is hyperconvex;*
(2) *$\rho\colon \pi_1(S) \to \mathrm{PSL}_n(\mathbb{R})$ is Hitchin if and only if $\rho\colon \Gamma \to \mathrm{PSL}_n(\mathbb{R})$ is Hitchin.*

Proof. Left to the reader as an exercise. □

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