GEAR 2013 summer workshop on Higher Teichmuller-Thurston theory

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SEMISIMPLE LIE GROUPS

BRIAN COLLIER

1. Outiline

The goal is to talk about semisimple Lie groups, mainly noncompact real semisimple Lie groups. This is a very broad subject so we will do our best to be concise and cover just what we need to discuss the Cartan decomposition theorem, parabolic subgroups and associated symmetric spaces. To do this we need to first introduce some notation and review a lot of Lie algebra theory, the sources I have used are [1] [3] [5] [4] [6]. Here is a general outline of what will be covered.

1. General set up
2. Real forms, Cartan involutions and Cartan Decomposition Theorem
3. Some facts about the symmetric space G/K.
4. Complex semisimple Lie algebras
5. Restricted roots, Iwasawa Decomposition and real parabolic subgroups

The talk will have many examples even though this write up does not. Very little will be proven.

2. General Set up

Let G be a Lie group, left translation gives rise to a finite dimensional Lie subalgebra of the Lie algebra of vector fields on G with Lie bracket, this the Lie algebra of left invariant vector fields. If $L_g : G \to G$ denotes left translation by $g \in G$ then a vector field $X \in (\Gamma(TG))$ is left invariant if $dL_g(X) = X$ for all $g \in G$, i.e. $\forall g \in G$ and $\forall\ h \in G$, we have $dL_g(X(h)) = X(g \cdot h)$.

The set of left invariant vector fields is closed under Lie bracket, making it into a Lie subalgebra which we denote by $\mathfrak{g} \subset \Gamma(TG)$. Since every left invariant vector field is determined by its value at a single point it follows that $\mathfrak{g} \cong T_gG$ for any $g \in G$, we therefore identify $\mathfrak{g}$ with the tangent space at the identity, $\mathfrak{g} = T_eG$.

The bracket on T_eG is defined by taking the Lie bracket of the associated left invariant vector fields and evaluating it at the identity. With this description of the Lie algebra we get a natural description of the exponential map, $exp : \mathfrak{g} \to G$ by flowing along the left invariant vector fields associated to $\mathfrak{g}$.

2.1. Lie algebras.

This section is mostly from [6]. Given a finite dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ we get a natural representation of

$$ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

defined by

$$ad_X(Y) = [X, Y]$$

(the fact that it is a representation follows from the Jacobi identity). The image of ad

$$ad_{\mathfrak{g}} \subset \mathfrak{Der}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$$

is, by definition, the ideal of inner derivations in the Lie algbera $\mathfrak{der}(\mathfrak{g})$ of derivations on $\mathfrak{g}$.

For every $X \in \mathfrak{g}$ we have $exp(ad_X) \subset Aut(\mathfrak{g}) \subset GL(\mathfrak{g})$, the image of exp in $Aut(\mathfrak{g})$ is called the group of inner automorphisms and will be denoted $Int(\mathfrak{g})$. In fact, the Lie subalgebras $ad_{\mathfrak{g}} \subset \mathfrak{der}(\mathfrak{g})$ are the tangent Lie algebras of $Int(\mathfrak{g}) \subset Aut(\mathfrak{g})$.

If G is a Lie group then G acts on itself via inner automorphisms. That is, if $\alpha_g : G \to G$ is defined by

$$\alpha_g(h) = g \cdot h \cdot g^{-1}$$

we get a representation $Ad : G \to Aut(\mathfrak{g}) \subset GL(\mathfrak{g})$ of G in $\mathfrak{g}$ given by $Ad_g = d(\alpha_g)(e)$. The Lie algebra representation corresponding to Ad (i.e. $d(Ad)(e)$) is the previously mentioned ad. The image of Ad, denoted Ad_G, is called the adjoint group of G. It is the Lie group with lie algebra $\mathfrak{g}$ and trivial center.

We will mainly be interested in semisimple Lie algebras and Lie groups. A Lie group is called nilpotent, solvable, reductive, semisimple or simple if its Lie algebra has this property. We begin with a few definitions.

Definition 2.1. A Lie algebra $\mathfrak{g}$ is simple if it is nonabelian and contains no nontrivial ideals.

Semisimple Lie algebras are direct sums of simple Lie algebras. This is usually something that is proven and not taken as a definition. To properly define semisimple Lie algebras we need to first define solvable Lie algebras.

Definition 2.2. Let $\mathfrak{g}$ be a Lie algebra, let $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$ and define $\mathfrak{g}^j = [\mathfrak{g}^{j-1}, \mathfrak{g}^{j-1}]$ inductively, $\mathfrak{g}$ is called solvable if the series

$$\mathfrak{g} \supset \mathfrak{g}^1 \supset \mathfrak{g}^2 \supset \dots$$

terminates.

The standard example of a solvable Lie algebra is the Lie algebra of upper triangular matrices.

Definition 2.3. Let $\mathfrak{g}$ be a Lie algebra, let $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$ and define $\mathfrak{g}_j = [\mathfrak{g}, \mathfrak{g}_{j-1}]$ inductively, $\mathfrak{g}$ is called nilpotent if the series

$$\mathfrak{g} \supset \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \dots$$

terminates.

The standard example of a nilpotent Lie algebra is the Lie algebra of strictly upper triangular matrices.

Definition 2.4. A Lie algebra $\mathfrak{g}$ is semisimple if it contains no nonzero solvable ideals.

Another useful criterion for semisimplicity was given by Cartan using the Killing form. The representation ad allows us to define an $Int(\mathfrak{g})$ invariant symmetric bilinear form called the Killing form,

$$B_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K},$$

defined by

$$B_{\mathfrak{g}}(X, Y) = Tr(ad_x ad_y).$$

Theorem 2.5. *Lie algebra is semisimple if and only if $B_{\mathfrak{g}}$ is nondegenerate.*

From this we see that semisimple Lie algebras have trivial centers. One key property of semisimple Lie algebra's is that every finite dimensional representation splits as a direct sum of irreducible subrepresentations. The converse is not true, for instance $\mathfrak{g}(n,\mathbb{R})$ has the property that every finite dimensional representation splits as a direct sum of irreducible subrepresentations; but it is not semisimple (it has a center). This leads us to the slightly more general notion of a reductive Lie algebra

Definition 2.6. A Lie algebra is reductive if it is of the form $\mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g},\mathfrak{g}]$. Alternatively, a Lie algebra is reductive if and only if every finite dimensional representation splits as a direct sum of irreducibles.

3. Real forms, Cartan involutions and Cartan Decomposition Theorem

This section is mostly from [5] and [6]. As mentioned at the beginning, we are mainly interested in real semisimple Lie groups and Lie algebra's. However, the structure theory of complex semisimple Lie algebras is simpler and has many nice properties. To go from complex Lie algbera's to real Lie algebras we need to discuss real forms. First, note that if $\mathfrak{g}$ is a real Lie algebra then

$$\mathfrak{g} \otimes \mathbb{C} \cong \mathfrak{g} \oplus i\mathfrak{g}$$

is a complex Lie algebra, note also that $\mathfrak{g}$ is semisimple if and only if $\mathfrak{g} \otimes \mathbb{C}$ is. In the decomposition

$$\mathfrak{g} \otimes \mathbb{C} = \mathfrak{g} \oplus i\mathfrak{g},$$

$\mathfrak{g}$ is the fixed points of the conjugate linear involution of conjugation, this leads us to the definition of a general real form.

Definition 3.1. Let $\mathfrak{g}$ be a complex Lie algebra and

$$\sigma : \mathfrak{g} \to \mathfrak{g}$$

a conjugate linear involution, σ defines a decomposition

$$\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$$

of $\mathfrak{g}$ into its $+1$ and -1 eigenspaces. The subalgebra $\mathfrak{g}^+$ is called a real form of $\mathfrak{g}$.

If $\mathfrak{g}^+ \subset \mathfrak{g}$ is the real form coming from involution σ then, since σ is conjugate linear, it has the property $\mathfrak{g}^+ \otimes \mathbb{C} \cong \mathfrak{g}$.

Here are some definitions and facts we will need:

1. A Lie algebra is called compact if and only if it is the Lie algebra of a compact Lie group.
2. It is a theorem that a Lie algebra is compact if and only if it admits an invariant scalar product.
3. In the semisimple case the Killing form, B, is negative definite if and only if the Lie algebra is compact, in which case $-B$ provides an invariant scalar product.
4. Complex semisimple Lie algebra's have two special types of real forms; a compact real form and a split real form both of which are unique up to conjugation.
5. The compact real form is a real form $\mathfrak{g}^+ \subset \mathfrak{g}$ that is a compact Lie algebra.

We will denote the compact real involution by σ, and denote the $+1$ eigenspace of σ by $\mathfrak{u}$. Then

$$\mathfrak{g} = \mathfrak{u} \oplus i\mathfrak{u},$$

and the Killing form is negative definite on $\mathfrak{u}$ and positive definite on $i\mathfrak{u}$. If G is a complex semisimple Lie group with Lie algebra $\mathfrak{g}$, then $exp(\mathfrak{u})$ is a maximal compact subgroup of the complex Lie group G.

3.1. **Cartan Decomposition on Lie algebra level.**

Definition 3.2. Let $\mathfrak{g}$ be a real Lie algebra, an involution $\theta : \mathfrak{g}\to\mathfrak{g}$ is called a Cartan involution if the symmetric bilinear form $B_\theta(X,Y) = -B(X,\theta(Y))$ is positive definite.

Every real semisimple Lie algebra has a Cartan involution, this is a consequence of the existence of compact real forms for complex semisimple Lie algebras. To see this, let $\mathfrak{g}$ be a real semisimple Lie algebra and

$$\mathfrak{g}^{\mathbb{C}} \cong \mathfrak{g} \oplus i\mathfrak{g}$$

be its complexification, let τ be the conjugation on $\mathfrak{g}^{\mathbb{C}}$, giving real form $\mathfrak{g}$. Let $\sigma : \mathfrak{g}^{\mathbb{C}}\to\mathfrak{g}^{\mathbb{C}}$ be a compact real form so that $\tau\sigma = \sigma\tau$, (such a compact form always exists).

Claim 3.3. *$(\tau\sigma) = \theta$ is a Cartan involution when restricted to $\mathfrak{g}$.*

Proof. The involution θ defines an eigenspace decomposition of $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}^{\mathbb{C}}_+ \oplus \mathfrak{g}^{\mathbb{C}}_-$. Since θ commutes with σ and τ, both real forms $\mathfrak{u}$ and $\mathfrak{g}$ are stable under θ. Thus we have the eigenspace decompositions

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$$
$$\mathfrak{u} = \mathfrak{u}_+ \oplus \mathfrak{u}_-.$$

Call $\mathfrak{g}_+ = \mathfrak{k}$ and $\mathfrak{g}_- = \mathfrak{p}$, note that θ coincides with σ on $\mathfrak{g}$, so $\mathfrak{k} = \mathfrak{u}_+$ and $\mathfrak{p} = i\mathfrak{u}_-$. We have the following decompositions:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$
$$\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}.$$

It is now straightforward to check that θ defines a Cartan involution. □

We call a choice of such a involution θ and its corresponding decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ a Cartan Decomposition.

Note for a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ we have

$$[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k} \qquad [\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p} \qquad [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$$

It follows that the Cartan decomposition is orthogonal with respect to both B and B_θ. The Killing form B is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$, thus $\mathfrak{k} \subset \mathfrak{g}$ is a maximal compact subalgebra. If G is a semisimple Lie group with Lie algebra $\mathfrak{g}$, then $\exp(\mathfrak{k}) = K \subset G$ is a maximal compact subgroup. Thus a Cartan decomposition gives a Ad_K invariant orthogonal splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

3.2. **Cartan Decomposition on Lie group level.**

The Cartan decomposition theorem gives a decomposition of a real semisimple Lie Group G with respect to a maximal compact subgroup. We will not prove the theorem, however during the talk we will give some examples.

Theorem 3.4. *Let G be a semisimple Lie group, let θ be a Cartan involution of $\mathfrak{g}$, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition, and let K be the analytic subgroup of G with Lie algebra $\mathfrak{k}$. Then*

1. *There exists a Lie group involutative automorphism Θ of G with differential θ.*
2. *The subgroup of G fixed by Θ is K*
3. *The mapping $K \times \mathfrak{p} \rightarrow G$ defined by $(k, X) \mapsto (kexp(X))$ is a diffeomorphism.*
4. *K is closed and contains the center of G.*

Note that this theorem generalizes the polar decomposition of invertible matrices. Notice also that this decomposition is not a decomposition into subgroups, as $exp(\mathfrak{p}) \subset G$ is not a subgroup.

4. The symmetric space G/K

This brief section is mostly from [1]. Let $H \subset G$ be a closed subgroup and suppose $\mathfrak{g}$ admits a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$

that is Ad_H invariant and orthogonal with respect to a G-invariant metric. By the results in the previous section, if $K \subset G$ is a maximal compact subgroup then $\mathfrak{g}$ admits such a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

The Lie group G is the total space of a principal H bundle $G \rightarrow G/H$. If

$$\omega^G_{MC} \in \Omega^1(G, \mathfrak{g})$$

is the (left) Maurer-Cartan form of G and

$$P_{\mathfrak{h}} : \mathfrak{g} \rightarrow \mathfrak{h}$$

is the projection onto $\mathfrak{h}$ then

$$P_{\mathfrak{h}} \circ \omega^G_{MC} = \theta^c \in \Omega^1(G, \mathfrak{h})$$

defines a canonical connection one form on the principal H bundle $G \rightarrow G/H$.

The tangent bundle TG/H is canonically isomorphic to $G \underset{Ad_H}{\times} \mathfrak{m}$. Since TG/H is an associated bundle of $G \rightarrow G/H$, the canonical connection one form θ^c induces a covariant derivative ∇^c on TG/H. Furthermore, by construction, all G invariant tensors on G/H are covariantly constant with respect to ∇^c. A G-invariant metric is such a tensor, hence ∇^c is automatically a metric connection for any G-invariant metric on G/H.

The question of whether ∇^c is the Levi-Civita connection of a G-invariant metric requires examining the torsion of ∇^c. It turns out that the torsion tensor $T(\nabla^c)$ vanishes if and only in

$$[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$$

that is, if and only if G/H is a symmetric space, see [1] for more details. As a result, a lot of the Riemannian geometry of symmetric spaces can be understood from the Lie theory.

By the Cartan decomposition theorem, G/K is a symmetric space. The Levi-Civita connection of any G invariant metric is therefore the one induced by the canonical connection on $G \rightarrow G/K$. One place this is useful (for higher Teichmüller Theory) is in understanding the harmonic metric equations associated to a reductive representation. If Σ is a closed surface, the harmonic map associated to a representation

$$\rho : \pi_1(\Sigma) \rightarrow G$$

in Corlette's theorem [2] is a map $h : \widetilde{\Sigma} \rightarrow G/K$.

5. Complex semisimple Lie algebras

This section is mostly from [3]. For this section $\mathfrak{g}$ will be a complex semisimple Lie algebra. One key property of such a $\mathfrak{g}$ is: there exists a maximally commutative subalgebra $\mathfrak{h} \subset \mathfrak{g}$ whose adjoint action on $\mathfrak{g}$ is diagonal and is unique up to conjugation. By diagonal, we mean the Lie algebra representation $ad : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{g})$ decomposes $\mathfrak{g}$ into a direct sum irreducible representations called weight spaces. A non zero weight for the adjoint action is called a root; the roots form a finite subset $\Delta \subset \mathfrak{h}^*$ and $\mathfrak{h}$ acts on itself with weight 0. We have

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha.$$

where $\mathfrak{g}_0 = \mathfrak{h}$. The spaces $\mathfrak{g}_\alpha$ are the root spaces, they are defined by

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} | ad_H(X) = \alpha(H)X \quad \forall H \in \mathfrak{h}\}.$$

Some important properties of the root space decomposition include:
1. $\mathfrak{h}(\mathbb{R}) = \{h \in \mathfrak{h} | \alpha(h) \in \mathbb{R} \quad \forall \alpha \in \Delta\}$ is a real form of $\mathfrak{h}$.
2. If $\alpha \in \Delta$ then $n\alpha \in \Delta$ if and only if $n = -1$.
3. If $\alpha \in \Delta$ then $dim(\mathfrak{g}_\alpha) = 1$.
4. If $\alpha, \beta \in \Delta$ then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, in particular $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$ and $< \mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}, [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] >$ form a subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.
5. $< \Delta >= \mathfrak{h}^*$
6. The Killing form satisfies $B(u, v) = 0$ for two root vectors u, v corresponding to roots $\alpha, \beta \in \Delta$ with $\alpha + \beta \neq 0$. We also have that $B|\mathfrak{h}$ and $B|(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$ are nondegenerate and $B|\mathfrak{h}(\mathbb{R})$ is real and positive definite.

The dimension of a Cartan subalgebra is called the rank of $\mathfrak{g}$. Property 2 tells us that if we choose a linear function $l : \mathfrak{h}^* \to \mathbb{R}$ with the property $l(\Delta) \neq 0$, we have a splitting of Δ into positive and negative roots, $\Delta = \Delta_+ \cup \Delta_-$, here

$$\Delta_+ = \{\alpha \in \Delta | l(\alpha > 0\}.$$

A positive root $\alpha \in \Delta_+$ is called simple if it can not be written as a linear combination (with positive coefficients) of elements of Δ_+, we denote the set of simple roots by $\Pi = \{\alpha_1, \ldots, \alpha_n\} \subset \Delta_+$. It is clear that all elements of Δ are linear combinations of elements of Π, thus, by property 5 above we know that $< \Pi >= \mathfrak{h}^*$. In fact Π forms a basis for $\mathfrak{h}^*$, and provide us with a canonical basis of

$$\mathfrak{g} =< h_{\alpha_i}, e_\alpha, \tilde{e}_\alpha > .$$

Here $h_{\alpha_i} \in \mathfrak{h}$ is obtained from the isomorphism $\mathfrak{h}^* \to \mathfrak{h}$ given by the Killing form and $e_\alpha \in \mathfrak{g}_\alpha$ and $\tilde{e}_\alpha \in \mathfrak{g}_{-\alpha}$ for $\alpha \in \Delta_+$, are defined so that

$$< e_\alpha, \tilde{e}_\alpha, [e_\alpha, \tilde{e}_\alpha] = h_\alpha > \cong \mathfrak{sl}(2, \mathbb{C}).$$

This leads us to the definitions of standard complex Borel and parabolic subalgebras.

Definition 5.1. The standard Borel subalgebra (or minimal parabolic) with respect to a choice Cartan subalgebra and positive roots, is defined by

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha.$$

A standard parabolic subalgebra is any subalgebra containing $\mathfrak{b}$, a parabolic subalgebra is determined by a choice of subset of $-\Pi$.

General Parabolic subalgebras are conjugates of standard ones and parabolic subgroups are just exponentials of parabolic subalgebras. We will discuss these more in the real case below.

6. Restricted roots, Iwasawa Decomposition and real parabolic subgroups

This section is mostly from [5]. The goal of this section is to define parabolic subgroups of real semisimple Lie groups. To do this we need to introduce restricted roots and the Iwasawa Decomposition. Parabolic subgroups are defined by the property that G/P is compact. The main complication is that for a real semisimple Lie algebra the maximal abelian subalgebra that acts diagonally on $\mathfrak{g}$ may be too small to complexify to be a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. For split real Lie groups, this complication disappears.

We start with a real semisimple Lie algebra $\mathfrak{g}$ with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $\mathfrak{a}$ be the maximal abelian subspace of $\mathfrak{p}$, (note $exp(\mathfrak{a}) \cong (R^+)^{dim \mathfrak{a}}$) since the adjoint of $X \in \mathfrak{g}$ with respect to the postive definite form B_θ satisfies $(ad_X)^* = -ad_{\theta X}$, we see that X is self adjoint if and only if $X \in \mathfrak{p}$. Thus the set $\{ad_H | H \in \mathfrak{a}\}$ is a commiting family of self adjoint transformations of $\mathfrak{g}$. From this we conclude $\mathfrak{g}$ is a direct sum of simultaneous eigenspaces, all the eigenvalues being real.

For $\lambda \in \mathfrak{a}^*$ we have

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} | ad_H(X) = \lambda(H)X \quad \forall H \in \mathfrak{a}\},$$

if $\lambda \neq 0$ and $\mathfrak{g}_\lambda \neq 0$ then λ is called a restricted root. Let Σ denote the set of restricted roots.

The Lie algebra $\mathfrak{g}$ admits an orthogonal decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda$ satisfying the following:

1. $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$
2. $\theta \mathfrak{g}_\lambda = \mathfrak{g}_{-\lambda}$ hence $\lambda \in \Sigma \implies -\lambda \in \Sigma$.
3. $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$ orthogonally, where $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$.

The dimension of $\mathfrak{a}$ is called the real rank of $\mathfrak{g}$, if the real rank is the same as the rank of $\mathfrak{g}$ then $\mathfrak{g}$ is a split real form. For split real forms the dimensions of $\mathfrak{g}_\lambda$ are all 1.

6.1. **Iwasawa Decomposition.** Choose a notion of positivity for the restricted roots Σ and let Σ^+ be the positive restricted roots. Define $\mathfrak{n} = \bigoplus_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda$, this is a nilpotent Lie algebra, the Iwasawa decomposition theorem on the level of Lie algebra's is the following.

Theorem 6.1. *In the notation above, $\mathfrak{g}$ is a vector space direct sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Here $\mathfrak{a}$ is abelian, $\mathfrak{n}$ is nilpotent, $\mathfrak{a} \oplus \mathfrak{n}$ is solvable subalgebra of $\mathfrak{g}$ and $[\mathfrak{a} \oplus \mathfrak{n}, \mathfrak{a} \oplus \mathfrak{n}] = \mathfrak{n}$.*

There is an analogous theorem on the level of Lie Groups.

6.2. **Parabolic subalgebras and subgroups.** A parabolic subgroup of a real semisimple Lie group is the exponential of a parabolic subalgebra, $Q \subset G$ is a parabolic subgroup if and only if G/Q is compact and $\mathfrak{q} \subset \mathfrak{g}$ is a parabolic subalgebra if and only if $\mathfrak{q}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$ is a parabolic subalgebra. Just as in the set up for complex semisimple Lie algebras, we can choose a set of positive restricted roots $\Sigma^+ \subset \Sigma$ and consider the corresponding simple restricted roots.

Definition 6.2. The mininmal parabolic subalgebra (with respect to the choices made) is the subalgebra

$$\mathfrak{b} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda.$$

A general minimal parabolic is one which is conjugate to $\mathfrak{b}$.

Definition 6.3. A parabolic subalgebra is a subalgebra containing a minimal parabolic.

Let $\Pi \subset \Sigma^+$ be the subset of simple restricted roots, up to conjugation the parabolic subalgebras of $\mathfrak{g}$ are characterized by subsets $\Pi' \subset \Pi$. To see this consider

$$\Gamma = \Sigma^+ \cup \{\beta \in \Sigma | \beta \in span(\Pi')\},$$

we get the following parabolic subalgebra from Γ

$$\mathfrak{q} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Gamma} \mathfrak{g}_\lambda.$$

By flipping our notion of positive to roots, (i.e. the negative roots before are now the positive roots) we have a similar minimal parabolic subalgebra

$$\mathfrak{b}^- = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Sigma^-} \mathfrak{g}_\lambda.$$

Similarly corresponding to $\Pi' \subset \Pi$ we have $-\Pi' \subset -\Pi$ and

$$-\Gamma = \Sigma^- \cup \{\beta \in \Sigma | \beta \in span(-\Pi')\},$$

and the corresponding parabolic subalgebra

$$\mathfrak{q}^- = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in -\Gamma} \mathfrak{g}_\lambda.$$

For notational clarity we write $\mathfrak{q} = \mathfrak{q}^+$ and $\mathfrak{b} = \mathfrak{b}^+$, the intersection

$$\mathfrak{q}^+ \cap \mathfrak{q}^- = \mathfrak{g}_0 \bigoplus_{\{\lambda \in \Sigma | \lambda \in span(\Pi')\}} \mathfrak{g}_\lambda$$

is called the Levi subalgebra of $\mathfrak{q}$. By exponentiating we get two parabolics $Q^+ = exp(\mathfrak{q}^+)$ and $Q^- = exp(\mathfrak{q}^-)$ called opposite parabolics, the intersection $Q^+ \cap Q^-$ is called a Levi subgroup associated to Q.

References

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Anosov-representations: Basic definitions and properties

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Abstract

These informal notes are a brief summary of the talk I gave, with the same title, at the Workshop on Higher Teichmüller-Thurston theory, which took place in Northport Maine in the last week of June 2013. Use at your own risk.

1 Informal introduction

Anosov representations were introduced by Labourie [Lab] and describe so-called 'Anosov structures'. The basic idea is that, rather than encoding a sort of geometric structure on your manifold, such representations encode a type of dynamical structure that is analogous to that of an Anosov flow. This turns out to be a more flexible type of structure that is applicable in a wider context.

After defining the concept, Labourie went on to establish various results about Anosov representations. In particular, he showed that representations in Hitchin components are Anosov and used this to extend the known results about Hitchin components. For example, he showed that the mapping class group acts property on the Hitchin component, and that all representations in the Hitchin component are discrete, faithful, and purely loxodromic.

Taking the idea behind Anosov representations further, Guichard and Wienhard [GW] expanded the notion from Anosov structures on manifolds to representations of arbitrary word hyperbolic groups and showed that the definition may be reformulated in terms of a pair of *Anosov maps* from the boundary of the word hyperbolic group into certain compact homogeneous spaces. Working in this framework, they showed, among other things, that the set of Anosov representations is open in the representation variety.

2 Motivation

Upon a first encounter with the topic, the definition of Anosov representations can appear quite opaque making it difficult to perceive the idea behind the concept. In attempts to circumvent this frustration, in this section I'll discuss a motivational picture that I hope will help illuminate the definition of Anosov representations in §3.

2.1 Anosov flows

Before getting into the specifics of Anosov representations, let us first recall the motivating picture of Anosov flows. For a more detailed discussion of Anosov flows and their properties, see the article [Pla] by Plante and the references therein.

Let M be a compact Riemannian manifold equipped with a C^1–flow $\phi_t\colon M \to M$. This is said to be an **Anosov flow** if there is a ϕ_t–invariant splitting

$$TM = E^s \oplus E^u \oplus E^T$$

of the tangent bundle TM into subbundles E^s, E^u, and E^T (so each bundle must be invariant under the derivative $D\phi_t$) that satisfy

1. E^T is a line bundle that is tangent to the flow ϕ_t,
2. E^u is *expanding*, that is, there exist constants $A > 0$, $\mu > 1$ so that for all $t \in \mathbb{R}$ and $v \in E^u$ we have
$$\|D\phi_t(v)\| \geq A\mu^t \|v\|,$$
3. E^s is *contracting*, that is, there exist constants $B > 0$, $\lambda < 1$ so that for all $t \in \mathbb{R}$ and $v \in E^s$ we have
$$\|D\phi_t(v)\| \leq B\lambda^t \|v\|.$$

(Above, the norms of vectors are calculated with respect to the Riemannian metric.) We remark that since M is compact, conditions (2) and (3) are independent of the chosen metric; thus being Anosov is truly a property of the flow itself.

In the case of an Anosov flow, it is known that the bundles E^u and E^s are in fact integrable, meaning that they are tangent to the leaves of C^1 foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ of M.

Example 2.1. The canonical example of an Anosov flow is the geodesic flow on (the unit tangent bundle of) a compact negatively curved Riemannian manifold. To briefly see why such a flow is Anosov, let us consider the case of a closed surface S equipped with a hyperbolic metric. Then the universal cover $\tilde{S}$ is identified with the Poincaré upper half plane $\mathbb{H}^2$ and the unit tangent bundle $T^1(\tilde{S})$ with $\mathrm{PSL}(2,\mathbb{R})$. For concreteness, let's identify $I \in \mathrm{PSL}(2,\mathbb{R})$ with the vertical tangent vector at the point $i \in \mathbb{H}^2$.

Now, the tangent space $T_I(\mathrm{PSL}(2,\mathbb{R}))$ is the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of 2×2 traceless matrices, which has a basis given by
$$G_I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \quad X_I = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \quad Y_I = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right).$$
These generate left-invariant vector fields G, X, and Y on $\mathrm{PSL}(2,\mathbb{R})$ (which thus descend to vector fields on $T^1(S)$) which induce a splitting of $T(\mathrm{PSL}(2,\mathbb{R}))$ as the sum of 3 line bundles E_G, E_X, and E_Y.

Exponentiating in the direction of these basis vectors gives the matrices
$$g_t := \exp(tG_I) = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}, \quad x_t := \exp(tX_I) = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right), \quad y_t := \exp(tY_I) = \left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right).$$
Multiplication on the *right* by g_t defines a flow on $\mathrm{PSL}(2,\mathbb{R})$ that is parallel to the vector field G (and similarly for x_t and y_t); in fact g_t is exactly the geodesic flow on $T^1(\mathbb{H}^2)$, and x_t and y_t are the horocycle flows. Since these flows commute with the left action on $\pi_1(S)$ on $\mathrm{PSL}(2,\mathbb{R})$, they descend to flows on $T^1(S)$.

With this concrete description of g_t, it is easy to check that the geodesic flow is Anosov with respect to the splitting $T(\mathrm{PSL}(2,\mathbb{R})) = E_G \oplus E_X \oplus E_Y$. Indeed, E_G is tangent to the flow, and the facts that $x_s g_t = e^{-t} g_t x_s$ and $y_s g_t = e^t g_t y_s$ show that the flow g_t is contracting and expanding on E_X and E_Y, respectively.

2.2 Philosophical ramblings

When studying representation spaces $\mathrm{Hom}(\Gamma, G)$, a goal is often to understand what sort of information these representations encode. For example, when $\Gamma = \pi_1(S)$ for a closed surface S and $G = \mathrm{PSL}(2,\mathbb{R})$, the discrete faithful representations in $\mathrm{Hom}(\Gamma, G)$ exactly encode hyperbolic structures on S. The question then arises, what sort of structure may be encoded by a representation into, for example, a larger Lie group?

We have seen in Example 2.1 that if a representation $\pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ defines a hyperbolic structure on S, then via the geodesic flow on $T^1(S)$ we actually have the extra structure of an Anosov system. In fact, as Example 2.1 illustrated, the distributions comprising the Anosov system in fact came from the Lie group $\mathrm{PSL}(2,\mathbb{R})$ itself. Now, if we change $\mathrm{PSL}(2,\mathbb{R})$ into a larger-dimensional Lie group G, it becomes quite difficult for a representation $\pi_1(S) \to G$ to encode a geometric structure on S—any 'developing map' from $\tilde{S}$ to the symmetric space G/K could not be a local homeomorphism because dimensions would not match up, and so we are left searching for other candidate G–spaces to model the desired geometry on S.

Labourie's insight in [Lab] was that, in spite of these difficulties, perhaps such a representation $\pi_1(S) \to G$ could still encode some sort of *dynamical* structure on S in the same way that we saw the structure of the Lie group $\mathrm{PSL}(2,\mathbb{R})$ lead to an Anosov flow on $T^1(S)$ in Example 2.1. This idea turned out to work, and the added dynamical structure allowed Labourie and others to prove new results about the representation spaces $\mathrm{Hom}(\pi_1(S), G)$.

3 Anosov representations

Let G be a semisimple Lie group, and let (P^+, P^-) be a pair of opposite parabolic subgroups (see the notes [Col] from Brian Collier's talk on semisimple Lie groups for the definition and properties of parabolic subgroups). Then the quotients $\mathcal{F}^\pm = G/P^\pm$ are compact homogeneous spaces (since $P^\pm$ are parabolic). We also consider the homogeneous space $\mathcal{X} = G/L$, where $L = P^+ \cap P^-$.

If we let G act diagonally on $\mathcal{F}^+ \times \mathcal{F}^-$, then L is exactly the stabilizer of (eP^+, eP^-) and so we may identify $\mathcal{X} = G/L$ with an orbit in $\mathcal{F}^+ \times \mathcal{F}^-$. In fact this is the unique open orbit. From the product structure on $\mathcal{F}^+ \times \mathcal{F}^-$, the orbit $\mathcal{X}$ inherits two G–invariant distributions $E^\pm$, where the vector space over a point $x = (x^+, x^-) \in \mathcal{X} \subset \mathcal{F}^+ \times \mathcal{F}^-$ is $E^\pm_x = T_{x^\pm}\mathcal{F}^\pm$. It follows that any $\mathcal{X}$–bundle also comes equipped with such a pair of distributions, which we again call $E^\pm$.

Example 3.1. What exactly are the spaces $G/P^\pm$ and G/L? Let's take a look at these in our favorite setting where $G = \mathrm{PSL}(2,\mathbb{R})$. In this case, the Borel subgroup B is the subgroup of upper triangular matrices. Thus there are no proper subgroups between G and B and we are forced to take $P^+ = B$ and P^- as the subgroup of lower triangular matrices.

Again identifying $I \in G$ with the vertical-pointing tangent vector at $i \in \mathbb{H}^2$, we find that $P^+ \leq G$ is exactly the set of vertical-pointing tangent vectors in $T^1(\mathbb{H}^2)$. Indeed, using the notation of Example 2.1, every element of P^+ may be expressed as a product $g_t x_s$, and both of these flows preserve the set of vertical tangent vectors. In particular we see that all elements of P^+ point towards the same boundary point in $\partial\mathbb{H}^2$ (namely ∞). Multiplying this picture by $g \in G$, we find that all vectors in gP^+ point towards the boundary point $g \cdot \infty \in \partial\mathbb{H}^2$. Thus we have a natural identification of G/P^+ with the circle $S^1 = \partial\mathbb{H}^2$. Similarly all vectors in P^- point away from $0 \in \partial\mathbb{H}^2$, and so we again have an identification of G/P^- with $\partial\mathbb{H}^2$. Finally, G/L may be identified with the space of oriented geodesic lines in $\mathbb{H}^2$, which may equivalently be realized as the set

$$\partial(\mathbb{H}^2)^{(2)} = \partial\mathbb{H}^2 \times \partial\mathbb{H}^2 \setminus \{(t,t) \mid t \in \partial\mathbb{H}^2\}.$$

This same basic picture holds for every rank–1 Lie group. That is, whenever G is a rank–1 semisimple Lie group, the only opposite parabolic subgroups are the Borel subgroup B^+ and its opposite B^-, and for both of these there is a natural identification of $G/B^\pm$ with the boundary of the symmetric space G/K. When G has higher rank, there are more possibilities for the opposite parabolic subgroups (P^+, P^-) and the quotients $G/P^\pm$ can be identified with certain *proper* subspaces of the boundary of the symmetric space G/K.

3.1 For Riemannian manifolds

Let N be a closed negatively curved Riemannian manifold, and let $M = T^1(N)$ be its unit tangent bundle. We then have the universal cover $\tilde{N}$ and its unit tangent bundle $\widehat{M} = T^1(\tilde{N})$, which is a $\Gamma = \pi_1(N)$–cover of M. Let ϕ_t denote the geodesic flow on $\widehat{M}$; this descends to the geodesic flow (also denoted ϕ_t) on M.

Now suppose that we have a representation $\rho\colon \Gamma \to G$. The product $\widehat{M} \times \mathcal{X}$ becomes a Γ–space, where Γ acts diagonally on $\widehat{M}$ by deck transformations and on $\mathcal{X}$ via the representation ρ. We denote the quotient under this action by

$$\mathcal{X}_\rho := \widehat{M} \times_\rho \mathcal{X} = \Gamma \backslash (\widehat{M} \times \mathcal{X}).$$

The projection of $\widehat{M} \times \mathcal{X}$ onto the first factor descends to a map $\mathcal{X}_\rho \to M$ giving $\mathcal{X}_\rho$ the structure of an $\mathcal{X}$–bundle over M. Moreover, there is a natural flat connection on the bundle $\widehat{M} \times \mathcal{X} \to \widehat{M}$ (coming from the product structure) and this gives us a flat connection on $\mathcal{X}_\rho$. (Recall that a **connection** on a bundle is simply a choice of a way to lift any smooth path in the base to the total space. **Flat** here just means that the holonomy is locally trivial, that is, each point in the base has a neighborhood on which all closed loops lift to *closed* loops in the total space). Additionally, we emphasize that $\mathcal{X}_\rho$ comes equipped with two vector bundles $E^\pm \to \mathcal{X}_\rho$ (in fact, these are distributions $E^\pm \leq T\mathcal{X}_\rho$) coming the aforementioned distributions $E^\pm$ on the fiber $\mathcal{X}$.

Now here is an important point: The flow ϕ_t on $\widehat{M}$ lifts to a flow on $\widehat{M} \times \mathcal{X}$ defined by $\psi_t(m, x) = (\phi_t m, x)$. Furthermore, this flow is invariant under the action of Γ, and so defines a flow (also called ψ_t) on $\mathcal{X}_\rho$ which lifts ϕ_t. Notice that the product structure gives us a splitting of the tangent bundle

$$T(\widehat{M} \times \mathcal{X}) = T\widehat{M} \oplus T\mathcal{X},$$

and that ψ_t preserves this splitting (meaning that the derivative $D\psi_t$ preserves each subbundle). In addition, $D\psi_t$ acts trivially on $T\mathcal{X}$, so the flow also preserves each of the distributions $E^\pm \leq T\mathcal{X}$. To summarize, the flow ϕ_t on M lifts to a flow ψ_t on $\mathcal{X}_\rho$ that preserves the two distributions $E^\pm$.

Definition 3.2 (Anosov representation)**.** In the situation described above, a representation $\rho\colon \pi_1(N) = \Gamma \to G$ is called (P^+, P^-)–**Anosov** if

1. the bundle $\mathcal{X}_\rho$ admits a section $\sigma\colon M \to \mathcal{X}_\rho$ that is flat along flow lines (i.e., thinking of a flowline as a path $h\colon [0, 1] \to M$, the composition of this with the section σ must agree with a lift of h to $\mathcal{X}_\rho$ determined by the flat connection), and

2. the lifted action of ϕ_t (which is determined by the action of ψ_t) on σ^*E^+ (resp. σ^*E^-) is expanding (resp. contracting) (see §2.1 for definitions of these terms).

Remark 3.3. In the definition of an Anosov flow (see §2.1), the expanding and contracting properties were measured with respect the Riemannian norm on each tangent bundle. In the setting of Definition 3.2 the bundles $\sigma^*E^\pm$ are not subbundles of the tangent bundle TM, so we don't use the Riemannian metric here. Rather, property (2) above just means that there is some continuous family of norms $(\|\cdot\|_m)_{m \in M}$ on the fibers of the bundles $\sigma^*E^\pm$ so that the expanding/contracting properties are satisfied. Again, since M is compact this does not depend on the choice of the family of norms.

3.2 The equivariant maps

Let us further study the structure of an Anosov representation $\rho\colon \Gamma \to G$, where as above $\Gamma = \pi_1(N)$. A section σ of the bundle $\mathcal{X}_\rho \to M$ is equivalent to a ρ–equivariant $\hat{\sigma}\colon \widehat{M} \to \mathcal{X}$, and it is easy to see that σ is 'flat along flow lines' if and only if $\hat{\sigma}$ is ϕ_t–equivariant. Such a map may instead be considered to be defined on the space

$$\widehat{M}/\phi_t = \partial\tilde{N}^{(2)} = \partial\tilde{N} \times \partial\tilde{N} \setminus \{(t, t) \mid t \in \partial\tilde{N}\}$$

of ordered pairs of distinct boundary points of $\tilde{N}$. Thus the section $\sigma\colon M \to \mathcal{X}_\rho$ in the definition of an Anosov representation is in fact equivalent to a pair of maps

$$\hat{\sigma} = (\xi^+, \xi^-)\colon \partial\tilde{N}^{(2)} \to \mathcal{X} \subset \mathcal{F}^+ \times \mathcal{F}^-.$$

Furthermore, the contracting property of the Anosov representation implies that $\xi^+\colon \partial\tilde{N}^{(2)} \to \mathcal{F}^+$ in fact only depends on the first coordinate, that is, it factors through the projection $\partial\tilde{N}^{(2)} \to \partial\tilde{N}$ onto the first factor. Similarly ξ^- factors through the projection onto the second factor. Thus from an Anosov representation we obtain a pair of ρ–equivariant maps

$$\xi^\pm\colon \partial\tilde{N} \to \mathcal{F}^\pm.$$

These maps are called the **Anosov maps** associated to the Anosov representation. Conversely, given such maps, one may recover the section $\sigma\colon M \to \mathcal{X}_\rho$, and the properties imposed on σ in Definition 3.2 can be reformulated in terms of these Anosov maps.

In summary, the definition of a (P^+, P^-)–Anosov representation $\rho\colon \pi_1(N) \to G$ may be reformulated in terms of the existence of continuous ρ–equivariant maps $\xi^\pm \partial\tilde{N} \to G/P^\pm$ satisfying certain properties which are analogous to those in Definition 3.2 above; see [GW, Proposition 2.7] for details.

3.3 For hyperbolic groups

Whereas Labourie originally formulated the notion of Anosov representations in the setting of negatively curved manifolds (as described in § 3.1 above), Guichard and Wienhard [GW] extended this notion to representations $\rho\colon \Gamma \to G$ of an arbitrary word hyperbolic group Γ.

The key in this definition is to replace the space $(\widehat{M}, \phi_t)$ with an appropriate 'flow space' for the hyperbolic group. Such a space, which we will denote $\widehat{\Gamma}$, was introduced by Gromov [Gro] and further developed by Champetier [Cha], Mineyev [Min] and others. This space comes equipped with an action by the group $\Gamma \times \mathbb{R} \rtimes (\mathbb{Z}/2\mathbb{Z})$, where we should think of Γ as acting by deck transformations, the $\mathbb{R}$ as a 'geodesic flow', and the $\mathbb{Z}/2\mathbb{Z}$ as reversing direction. In particular, every orbit $\Gamma \to \widehat{\Gamma}$ is a quasi-isometry, the $\mathbb{R}$–orbits $\mathbb{R} \to \widehat{\Gamma}$ are quasi-geodesics, and the induced map

$$\widehat{\Gamma}/\mathbb{R} \to \partial\widehat{\Gamma}^{(2)} \cong \partial\Gamma^{(2)} = \partial\Gamma \times \partial\Gamma \setminus \{(t,t) \mid t \in \partial\Gamma\}$$

is a homeomorphism.

While one does not have all the nice manifold machinery in this setting, given a representation $\rho\colon \Gamma \to G$, one can nevertheless consider either ρ–equivariant maps $\xi^\pm\colon \partial\Gamma \to \mathcal{F}^\pm$ or sections of the bundle $\mathcal{X}_\rho = \widehat{\Gamma} \times_\rho \mathcal{X} \to \Gamma\backslash\widehat{\Gamma}$ that are flat along flow lines. In either of these settings, one can then formulate the criterion for being (P^+, P^-)–Anosov in terms of expanding/contracting properties that are directly analogous to those considered in §§3.1–3.2 above. For details, see §2.3 of [GW].

4 L–Cartan projections

Given a hyperbolic group Γ and a representation $\rho\colon \Gamma \to G$, we again form the associated flat bundle $\mathcal{X}_\rho = \widehat{\Gamma} \times_\rho \mathcal{X}$. When attempting to verify whether ρ satisfies the definition of an Anosov representation, it is often difficult to check the expanding/contracting properties because any candidate section $\sigma\colon \Gamma\backslash\widehat{\Gamma} \to \mathcal{X}_\rho$ must be flat, and therefore effectively 'constant', along flowlines. To aid in this task, Guichard and Wienhard introduce the tool of L–Cartan projections:

Recall that up to conjugacy, every pair of opposite parabolic subgroups is determined by choosing some subset of the simple roots in a root system. In particular there are not so many choices for the parabolic subgroups $P^\pm$. Going forward, let's focus on the parabolic subgroups $P^\pm = P^\pm_\Theta$ determined by such a subset Θ of the simple roots Δ.

Recall that $L = P^+ \cap P^-$ is the common Levi component of our parabolic subgroups. We consider its Weyl chamber $\mathfrak{a}_L^+$, which can be viewed as a subalgebra of the Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{g}$. Specifically, $\mathfrak{a}_L^+ = \{a \in \mathfrak{a} \mid \alpha(a) > 0, \text{ for all } \alpha \in \Theta\}$ (recall Θ is a subset of the dual Lie algebra $\mathfrak{a}^*$).

Let M be the maximal compact subgroup of L, and set $\mathcal{Y} = G/M$. Given our representation $\rho\colon \Gamma \to G$, we can then form the flat bundle $\mathcal{Y}_\rho = \widehat{\Gamma} \times_\rho \mathcal{Y}$, which is an L/M–bundle over $\mathcal{X}_\rho$. By lifting a candidate section $\sigma\colon \Gamma\backslash\widehat{\Gamma} \to \mathcal{X}_\rho$ to this larger bundle, one obtains a refined section $\beta\colon \Gamma\backslash\widehat{\Gamma} \to \mathcal{Y}_\rho$ which is *no longer* 'constant' along flowlines. Because the composition of the flowline through $m \in \Gamma\backslash\widehat{\Gamma}$ with β actually moves in the fiber $\mathcal{Y} = G/M$, one may look at the *change* between $\beta(m)$ and $\beta(\phi_t m)$ and measure the Cartan projection of this change. This projection is naturally an element of the Weyl chamber $\mathfrak{a}_L^+$. In this way, one obtains maps

$$\mu_\pm\colon \widehat{\Gamma} \times \mathbb{R} \to \overline{\mathfrak{a}}_L^+$$

which are termed the L–**Cartan projections** of σ. Then for $(m,t) \in (\Gamma\backslash\widehat{\Gamma}) \times \mathbb{R}$, one may define

$$A_\pm(m,t) = \min_{\alpha \in \Delta\setminus\Theta} \alpha(\mu_\pm(\hat{m},t)).$$

The key here is that these numbers really aren't all that complicated or too difficult to work with. Using these tools, the process of checking for an Anosov representation reduces to checking a few eigenvalues:

Proposition 4.1 (Prop 3.16 of [GW])**.** *Let $\rho\colon \Gamma \to G$ be a representation, let σ be a section of $\mathcal{X}_\rho$ which is flat along $\mathbb{R}$–orbits, and let $A_\pm$ be as above. The following are equivalent:*

1. *σ is an Anosov section (and so ρ is a (P^+, P^-)–Anosov representation).*
2. *There exist positive constants C, c such that for all $t \geq 0$ and $m \in \Gamma\backslash\widehat{\Gamma}$ one has $A_+(m,t) \geq ct - C$ and $A_-(m,-t) \geq ct - C$.*
3. *There exist positive constants C, c such that for all $t \geq 0$ and $m \in \Gamma\backslash\widehat{\Gamma}$ one has $A_+(m,t) \geq ct - C$.*
4. *$\lim_{t\to\infty} \inf_{m\in\Gamma\backslash\widehat{\Gamma}} A_+(m,t) = \infty$.*

5 Basic properties

One can show that if you are given an Anosov representation $\rho\colon \Gamma \to G$, then the corresponding Anosov maps $\xi^\pm\colon \partial\Gamma \to \mathcal{F}^\pm$ are *unique* (that is, there is just one choice of section $\sigma\colon \Gamma\backslash\widehat{\Gamma} \to \mathcal{X}_\rho$ that will be flat along flow lines and satisfy the expanding/contracting properties).

This uniqueness, together with the density of fixed points of non-torsion $\gamma \in \Gamma$ in $\partial\Gamma$ imply that a representation $\rho\colon \Gamma \to G$ is Anosov if and only if the same holds for the restriction to a finite index subgroup $\Gamma' < \Gamma$, or for the composition $\pi \circ \rho$ with a covering $\pi\colon \widehat{G} \to G$ of Lie groups (where you choose the parabolic subgroups compatibly).

Here are some other basic properties of Anosov representations:

- Every Anosov representation $\rho\colon \Gamma \to G$ is a quasi-isometric embedding; in particular:
  - $\ker(\rho)$ is finite,
  - $\rho(\Gamma) \leq G$ is discrete, and
  - ρ is well-displacing (meaning that γ and $\rho(\gamma)$ have uniformly comparable translation lengths)
- For a fixed parabolic subgroup $P < G$, the set of P–Anosov representations in $\mathrm{Hom}(\Gamma, G)$ is open.
- When G has rank one, there is exactly one conjugacy class of parabolic subgroups $P < G$ (so we can unambiguously talk about a representation begin Anosov), and the following are equivalent for a representation $\rho\colon \Gamma \to G$:
  - ρ is Anosov
  - there exits a continuous, equivariant, and injective map $\xi\colon \partial\Gamma \to G/P$
  - ρ is a quasi-isometry
  - $\ker(\rho)$ is finite and $\rho(\Gamma)$ is convex convex cocompact (i.e., acts cocompactly on some convex subset of the symmetric space G/K).

Finally, various well-studied types of representations are Anosov, such as maximal representations and representations in the Hitchin component.

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DOMAINS OF DISCONTINUITY FOR ANOSOV REPRESENTATIONS

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This article is a summary of the content in Section 7 to 10 of [3]. Let Γ be a word-hyperbolic group. The goal is to construct domains of discontinuity for a given Anosov representation ρ from Γ to a semisimple Lie group G. More specifically, we want to construct an open, Γ-invariant subset Ω in a compact G-space on which Γ acts properly discontinuously and cocompactly.

The general strategy for this construction is the following. In Section 2, we first construct domains of discontinuity for representations of Γ into automorphism groups of non-degenerate sesquilinear forms that are Anosov with respect to a stabilizer of an isotropic line or a maximal isotropic subspace. Using this, we can construct domains of discontinuity for Anosov representations of Γ into $SL(n, K)$, where $K = \mathbb{R}$ or $\mathbb{C}$. This is described in Section 3. In Section 4 we demonstrate how to generalize this construction for Anosov representations into a semisimple Lie group. Finally, we end the article by mentioning in Section 5 some structural results about these domains of discontinuity.

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1. Basics

We will start by giving a quick review of the general structure theory of semisimple Lie groups. The main purpose of this review is to establish notation that will be used in the rest of this article. For more details, one can refer to Chapter 2 of [2]. A brief description of this structure theory is also given in Section 3 of [3]. Let G be a semisimple Lie group and let $\mathfrak{g}$ be its Lie algebra. Choose a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$. This gives us a root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Lambda} \mathfrak{g}_\alpha,$$

where Λ is the set of roots for $\mathfrak{a}$. By choosing a Weyl chamber of this root space decomposition to be the positive Weyl chamber, we get a partition of Λ into the set of positive roots, denoted Λ^+, and the set of negative roots, denoted Λ^-. This also gives us a set of simple roots, denoted $\Delta \subset \Lambda^+$. Define

$$A := \exp(\mathfrak{a}),$$

$$\mathfrak{n} := \bigoplus_{\alpha \in \Lambda^+} \mathfrak{g}_\alpha$$

$$N := \exp(\mathfrak{n}),$$

and let K be the maximal compact subgroup of G with Lie algebra $\mathfrak{k}$.

Now, given any subset $\Theta \subset \Delta$, we can define a parabolic subgroup $P_\Theta := M_\Theta AN$, where M_Θ is the subgroup of K that fixes every element (via the adjoint representation) in $\bigcap_{\alpha\in\Theta}\ker(\alpha)$. The Lie algebra of P_Θ has a decomposition

$$\mathfrak{p}_\Theta = \bigoplus_{\alpha\in\Lambda^+\cup\Lambda_\Theta\cup\{0\}} \mathfrak{g}_\alpha,$$

where Λ_Θ is the set of roots that lie in the $\mathbb{R}$-span of the simple roots in Θ. It is known that any parabolic subgroup of G is conjugate to a parabolic subgroup P_Θ for some Θ.

2. Automorphism groups of non-degenerate sesquilinear forms

We will now describe the construction of domains of discontinuity for a very special type of representation $\rho : \Gamma \to G$.

Notation 2.1. Let (V, F) be a vector space over either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, equipped with a non-degenerate quadratic form F such that

(1) F is either an indefinite symmetric form or a skew-symmetric form when V is over $\mathbb{R}$,
(2) F is either a symmetric form, a skew-symmetric form or an indefinite Hermitian form when V is over $\mathbb{C}$,
(3) F is either an indefinite Hermitian form or a skew-hermitian form when V is over $\mathbb{H}$.

Denote the group of automorphisms of (V, F) by $G_F(V)$. Also, let $\mathcal{F}_0$ be the set of isotropic lines in V and let $\mathcal{F}_1$ be the set of maximal isotropic subspaces in V.

One can check that $G_F(V)$ acts transitively on both $\mathcal{F}_0$ and $\mathcal{F}_1$. Moreover, the stabilizers of isotropic lines and maximal isotropic subspaces in $G_F(V)$ are parabolic subgroups. In fact, we can describe what these parabolic subgroups are. The isotropic line stabilizers are conjugate to the parabolic subgroup $P_{\Delta\setminus\alpha_0}$ for some simple root α_0, and the parabolic subgroup that stabilizes a maximal isotropic subspace is conjugate to $P_{\Delta\setminus\alpha_1}$ for some simple root α_1. The roots α_0 and α_1 can be explicitly described, but we will not do so here. (See Section 7.2 of [3].) To simplify notation, we will denote $P_{\Delta\setminus\alpha_i}$ as Q_i. Using the orbit-stabilizer theorem, we can thus identify $\mathcal{F}_i$ with G/Q_i. In this section, we will only consider representations $\rho : \Gamma \to G$ that are Q_i-Anosov for some $i = 0, 1$.

Exercise 2.2. Work out what α_0 and α_1 are when G is $O(2, 3)$, $O(3, 3)$ and $Sp(4, \mathbb{R})$.

Let $\mathcal{F}_{01} := \{(D, P) \in \mathcal{F}_0 \times \mathcal{F}_1 : D \subset P\}$, and let $\pi_i : \mathcal{F}_{01} \to \mathcal{F}_i$ be the obvious projection. For any subset $A \subset \mathcal{F}_i$, let $K_A := \pi_{1-i}(\pi_i^{-1}(A))$, and observe that if A is closed in $\mathcal{F}_i$, then K_A is closed in $\mathcal{F}_{1-i}$. In particular, if $\rho : \Gamma \to G$ is Q_i-Anosov, then we have a Γ-invariant Anosov map $\xi : \partial_\infty\Gamma \to \mathcal{F}_i$ whose image is closed in $\mathcal{F}_i$, so $K_{\xi(\partial_\infty\Gamma)}$ is closed in $\mathcal{F}_{1-i}$. Let $\Omega_\rho := \mathcal{F}_{1-i} \setminus K_{\xi(\partial_\infty\Gamma)}$, and observe that Ω_ρ is an open, Γ-invariant subset of $\mathcal{F}_{1-i}$. In [3], Guichard and Wienhard proved the following theorem.

Theorem 2.3. *(Theorem 8.6 of* [3]*) Let $\rho : \Gamma \to G$ be a Q_i-Anosov representation. If Ω_ρ is nonempty, then Γ acts properly discontinuously and cocompactly on Ω_ρ.*

We will now give a vague description of their proof. To prove the proper discontinuity of the Γ action, they showed that the Q_i-Anosovness of ρ implies that its image satisfies some proximality conditions, which gives us some control of the Γ action on its limit set in $\mathcal{F}_i$. This then gives us enough control of the Γ action on $K_{\xi(\partial_\infty\Gamma)}$ to prove that Γ acts properly discontinuously on the complement. For the cocompactness of the action, they considered the orientable double cover $\mathcal{F}_i^{or}$ of $\mathcal{F}_i$. By making some suitable arrangements, they could assume without loss of generality that Γ is torsion free and the codimension of $K^{or}_{\xi(\partial_\infty\Gamma)}$ in $\mathcal{F}^{or}_{1-i}$ is sufficiently big. Hence, $\Gamma\backslash\Omega_\rho^{or}$ is a connected orientable manifold. Proving the cocompactness of the action of Γ in Ω_ρ is equivalent to proving the compactness of $\Gamma\backslash\Omega_\rho^{or}$, which in turn is equivalent to proving that $H^0_c(\Gamma\backslash\Omega^{or}_\rho)$ (cohomology with compact support and coefficients in $\mathbb{R}$) is non-zero. This can be done using Poincaré duality and long exact sequences of homology groups.

In the course of proving Theorem 2.3, several important properties of $K_{\xi(\partial_\infty\Gamma)}$ are used. The two most important ones are listed here.

Proposition 2.4. *(Proposition 8.1 of [3]) Let $\rho:\Gamma\to G$ be a Q_i-Anosov representation with associated Anosov map $\xi:\partial_\infty\Gamma\to\mathcal{F}_i$. Then $\pi_{1-i}:\pi_i^{-1}(\xi(\partial_\infty\Gamma))\to K_{\xi(\partial_\infty\Gamma)}$ is a homeomorphism.*

This proposition is a consequence of the transversality property of the Anosov map ξ. Also, it follows from this proposition that $K_{\xi(\partial_\infty\Gamma)}\simeq\pi_i^{-1}(\xi(\partial_\infty\Gamma))$ is a locally trivial fiber bundle over $\xi(\partial_\infty\Gamma)\simeq\partial_\infty\Gamma$. The fiber over a point $t\in\partial_\infty\Gamma$ is the set of lines through the origin in $\xi(t)$ if $i=1$, and is the set of maximal isotropic subspaces containing $\xi(t)$ if $i=0$.

The next proposition computes the "codimension" of $\partial_\infty\Gamma$ in $\mathcal{F}_{1-i}$. Here, the dimensions of the topological spaces are cohomological dimensions for Čech cohomology.

Proposition 2.5. *(Proposition 8.3 of [3]) Let $\rho:\Gamma\to G$ be a Q_i-Anosov representation. Let $\operatorname{vcd}(\Gamma)$ be the virtual cohomological dimension of Γ and let $\delta:=\dim(\mathcal{F}_{i-1})-\dim(K_{\xi(\partial_\infty\Gamma)})$. Then*

- *If $G=O(p,q)$, $U(p,q)$ or $Sp(p,q)$ (with $0<p\le q$), then $\delta=q-\operatorname{vcd}(\Gamma)$, $2q-\operatorname{vcd}(\Gamma)$ or $4q-\operatorname{vcd}(\Gamma)$ respectively.*
- *If $G=O(2n,\mathbb{C})$ or $O(2n-1,\mathbb{C})$, then $\delta=2n-\operatorname{vcd}(\Gamma)$.*
- *If $G=Sp(2n,\mathbb{R})$ or $Sp(2n,\mathbb{C})$, then $\delta=n+1-\operatorname{vcd}(\Gamma)$ or $2n+2-\operatorname{vcd}(\Gamma)$ respectively.*
- *If $G=SO^*(2n)$, then $\delta=4n-2-\operatorname{vcd}(\Gamma)$.*

In particular, if $\delta>0$ then Ω_ρ is non-empty. To prove Proposition 2.5, we use Proposition 2.4 to get that $\dim(K_{\xi(\partial_\infty\Gamma)})=\dim(M)+\dim(\partial_\infty\Gamma)$, where M is a fiber π_{1-i}. It is well-known result by Bestvina and Mess (see Corollary 1.4 of [1]) that $\dim(\partial_\infty\Gamma)=\operatorname{vcd}(\Gamma)-1$. Thus, computing $\dim(M)$ in each of the cases will give us the statements in Propositon 2.5.

Exercise 2.6. Explicitly compute δ for a Q_0-Anosov representation of the fundamental group of a closed surface with genus at least 2 into $O(2,3)$.

3. General strategy and the $SL(n,K)$ case

We would like to use the domains of discontinuity constructed in Section 2 to construct domains of discontinuity for Anosov representations into a general semisimple

Lie group. The strategy to do so is the following. Let G be an arbitrary semisimple Lie group and let $\rho : \Gamma \to G$ be a representation. We want to find a group homomorphism $\phi : G \to G_F(V)$ such that

(I) $\phi \circ \rho : \Gamma \to G_F(V)$ is Q_0-Anosov (or Q_1-Anosov),
(II) There is a maximal isotropic subspace T (or an isotropic line D resp.) in V whose stabilizer in G is a proper subgroup of G.

Suppose that we have such a group homomorphism ϕ, then we can construct a domain of discontinuity for ρ in the G-space G/AN as follows. (We only describe the construction in the Q_0-Anosov case. The Q_1-Anosov case is similar.)

We can define the ϕ-equivariant injective map

$$\begin{aligned} \phi_0 :& G/\operatorname{Stab}_G(T) \to \mathcal{F}_0(V) = G_F(V)/Q_0 \\ & g \cdot \operatorname{Stab}_G(T) \mapsto \phi(g) \cdot D \end{aligned} \tag{3.1}$$

Moreover, condition (I) implies that we have an Anosov map $\xi_{\phi\circ\rho} : \partial_\infty\Gamma \to \mathcal{F}_0(V)$ for $\phi \circ \rho$. By Section 2, we have a domain of discontinuity $\Omega_{\phi\circ\rho}$ in $\mathcal{F}_1(V)$ for $\phi \circ \rho$. One can then check that $\Omega_{\rho,V,T} := \phi_1^{-1}(\Omega_{\phi\circ\rho})$ is an open subset of G/AN that is stabilized by Γ, and on which Γ acts properly discontinuously and cocompactly. This is the required domain of discontinuity.

Thus, we now only need to construct the homomorphism ϕ with the required properties. The basic tool used to verify if ϕ satisfies the condition (1) is the following proposition.

Proposition 3.1. *(Proposition 4.4 of* [3]*) Let $\phi : G \to G'$ be a Lie group homomorphism. Let $\mathfrak{a}^+$, $\mathfrak{a}'^+$ be the positive Weyl chambers for G and G' respectively, and let Δ, Δ' be the corresponding set of simple roots for G and G' respectively. Also, let W' be the Weyl group for G'. For any $\Theta' \subset \Delta'$, and let $W'_{\Theta'}$ be the Weyl group of $L'_{\Theta'}$. Let $\Theta \subset \Delta$ and suppose that there exist w' in W' and $\Theta' \subset \Delta'$ such that*

$$\phi_*(\overline{\mathfrak{a}^+} \setminus \bigcup_{\alpha\in\Theta} \ker(\alpha)) \subset w' \cdot W'_{\Theta'} \cdot (\overline{\mathfrak{a}'^+} \setminus \bigcup_{\alpha'\in\Theta'} \ker(\alpha')).$$

Then for any P^+_Θ-Anosov representation $\rho : \Gamma \to G$, the representation $\phi \circ \rho$ is ${P'}^+_{\Theta'}$-Anosov.

The idea behind the proof of this proposition is to convert the contraction properties in the definition of an Anosov representation into more Lie group theoretic conditions, so that it is easy to see whether the contraction is preserved when we compose the Anosov representation ρ with a group homomorphism ϕ. Guichard and Wienhard did this using what they call L-Cartan projections. For more details, see Section 3.3 of [3].

Using Proposition 3.1 and some standard representation techniques, we can now give explicit constructions (See Section 10 of [3]) of domains of discontinuity for all Anosov representations into $SL(n, K)$, where K is either $\mathbb{R}$ or $\mathbb{C}$. We will briefly describe how this is done.

From the definition of a P-Anosov representation, it is easy to see that every P-Anosov representation is also $P_{\Theta_1} \cap P_{\Theta_2}$-Anosov, where $P_{\Theta_1}, P_{\Theta_2}$ is some pair of maximal parabolic subgroups of G such that the opposite of P_{Θ_1} is conjugate to P_{Θ_2}, and both P_{Θ_1} and P_{Θ_2} contain N. If we choose $\mathfrak{a}$ to be the set of diagonal matrices in $\mathfrak{sl}(n, K)$ and $\mathfrak{n}$ to be the upper triangular matrices in $\mathfrak{sl}(n, K)$, then in the standard basis, there is some k such that P_{Θ_1} is the stabilizer of the subspace spanned by the first k basis vectors and P_{Θ_2} is the stabilizer of the subspace spanned by the first $n-k$ basis vectors. In this case, it is clear that the natural map $\phi : SL(n,K) \to O(\mathrm{End}(\bigwedge^k K^n), tr)$ satisfies condition (I), and we can check that it satisfies condition (II) using Proposition 3.1. Thus, we can construct domains of discontinuity for P-Anosov representations to $SL(n,K)$ for any proper parabolic subgroup P in $SL(n,K)$.

Exercise 3.2. Consider the case when $G = SL(5,\mathbb{R})$. List all the possible parabolic subgroups of the form $P_{\Theta_1} \cap P_{\Theta_2}$ as described above. Also, give explicit descriptions of the domains of discontinuity for $P_{\Theta_1} \cap P_{\Theta_2}$-Anosov representations for all possible $P_{\Theta_1} \cap P_{\Theta_2}$.

4. Generalization to other semisimple Lie groups

To further extend this to the case of a general semisimple Lie group G, we use the following proposition.

Proposition 4.1. *(Proposition 4.3 of* [3]*) Let $\phi : G \to SL(V)$ be an irreducible finite dimensional linear representation of G. Let $V = D \oplus H$ be a decomposition of V into a line and a hyperplane, and set $Q_0^+ = \mathrm{Stab}_{SL(V)}(D)$, $Q_0^- = \mathrm{Stab}_{SL(V)}(H)$. Suppose that $(P^+, P^-) = (\mathrm{Stab}_G(D), \mathrm{Stab}_G(H))$ is a pair of opposite parabolic subgroups. Then a representation $\rho : \Gamma \to G$ is (P^+, P^-)-Anosov if and only if $\phi \circ \rho : \Gamma \to SL(V)$ is (Q_0^+, Q_0^-)-Anosov.*

This is in fact a specialization of Proposition 3.1 to the case when ϕ is irreducible and $G' = SL(V)$ for some finite dimensional vector space V.

To use this proposition, we need to build, for any semisimple Lie group G and any proper parabolic subgroup $P \subset G$, an irreducible representation $\phi : G \to SL(V)$ such that $P = \mathrm{Stab}_G(D)$ for some line D in V. This is not difficult to do using some standard representation theory. For instance, one can simply take $V = \bigwedge^p \mathfrak{g}$, where $\mathfrak{g}$ is the lie algebra of G and p is the dimension of the Lie algebra $\mathfrak{p}$ of P.

Now, given a P-Anosov representation $\rho : \Gamma \to G$, where G is a semisimple Lie group and P is a proper parabolic subgroup of G, we first construct an irreducible representation $\phi_1 : G \to SL(V)$ satisfying the conditions in Proposition 4.1. Then $\phi_1 \circ \rho$ is Anosov with respect to a line stabilizer, which is a proper parabolic subgroup of $SL(V)$. By the last paragraph of Section 3, we can then construct a representation $\phi_2 : SL(V) \to O(\mathrm{End}(K^n), tr)$ so that $\phi_2 \circ \phi_1 \circ \rho$ is Anosov with respect to the stabilizer of an isotropic line in $(\mathrm{End}(K^n), tr)$. Here, $K = \mathbb{C}$ if V is a complex vector space and $K = \mathbb{R}$ if V is a real vector space. Then one can check that $\phi = \phi_2 \circ \phi_1$ satisfies conditions (I) and (II), so we can use ϕ in the general strategy given at the start of Section 3 to obtain a domain of discontinuity for ρ.

5. Structure of the domain of discontinuity

Guichard and Wienhard also managed to obtain several results about the structure of these domains of discontinuity. We will end this article by mentioning two such results.

The first is an attempt to generalize Proposition 2.5 to the setting when G is an arbitrary semisimple Lie group. This is difficult to do in general, but in the cases where the virtual cohomological dimension of Γ is one or two, there are some concrete conditions to determine when the domain of discontinuity we constructed is nonempty. The main theorem in this direction is the following.

Theorem 5.1. *(Theorem 9.10 of* [3]*) Let $\rho : \Gamma \to G$ be a P-Anosov representation for which we can construct a domain of discontinuity, $\Omega_{\rho,V,T} \subset G/AN$.*

(1) If $\operatorname{vcd}(\Gamma) = 1$, then $\Omega_{\rho,V,T}$ is non-empty.
(2) If $\operatorname{vcd}(\Gamma) = 2$ and P contains every factor of G that is locally isomorphic to $SL(2,\mathbb{R})$, then $\Omega_{\rho,V,T}$ is nonempty.

The second result is about the topology of the quotient $\Gamma\backslash\Omega_{\rho,V,T}$. In general, it is not easy to determine the topology of $\Gamma\backslash\Omega_{\rho,V,T}$, but we have the following theorem.

Theorem 5.2. *(Theorem 9.12 of* [3]*) Let $\rho : \Gamma \to G$ be a P-Anosov representation for which we can construct a domain of discontinuity $\Omega_{\rho,V,T} \subset G/AN$. Let $\mathrm{Hom}_{P\text{-}Anosov}(\Gamma, G)$ be the space of P-Anosov homomorphisms from Γ to G. Then there exists an open neighborhood U of ρ in $\mathrm{Hom}_{P\text{-}Anosov}(\Gamma, G)$ such that*

$$\Gamma\backslash \bigcup_{\rho' \in U} \Omega_{\rho',V,T} \simeq U \times (\Gamma\backslash\Omega_{\rho,V,T})$$

as bundles over U.

In particular, the topology of $\Gamma\backslash\Omega_{\rho,V,T}$ depends only on the connected component of $\mathrm{Hom}_{\text{P-Anosov}}(\Gamma, G)$ containing ρ.

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HARMONIC MAPS - FROM REPRESENTATIONS TO HIGGS BUNDLES

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Abstract. The theory of Higgs bundles, initiated by Hitchin, has been instrumental in understanding the topology and geometry of character varieties. In addition, this gauge theoretic viewpoint provides a wealth of revealing, and puzzling, extra structure on the character variety. The key to the correspondence between representations of surface groups and Higgs bundles is the existence of an equivariant harmonic map from the universal cover of the surface to the symmetric space associated to a reductive Lie group G. The purpose of this note is to introduce the theory of Higgs bundles, with a strong emphasis put on the role of harmonic maps. We will begin with an overview of Fricke-Teichmüller theory via harmonic maps, and then explain how Higgs bundles generalize this point of view from $\mathrm{PSL}(2,\mathbb{R})$ to other Lie groups. We will focus exclusively on the linear groups $\mathrm{SL}(n,\mathbb{C})$, though it should be noted that the theory proceeds in an analogous fashion for any reductive Lie group.

Throughout, Σ is a fixed smooth, oriented, closed surface of genus greater than one. A base point $x \in \Sigma$ is fixed and let $\pi := \pi_1(\Sigma, x)$.

1. Fricke-Teichmüller theory via harmonic maps

Select hyperbolic metrics h_0 and h on Σ. In local isothermal coordinates $z = x^1 + ix^2$ and $w = y^1 + iy^2$ for h_0 and h respectively,

$$h_0 = e^{2u}|dz|^2$$

and

$$h = e^{2v}|dw|^2.$$

For any C^1-mapping $f : \Sigma \to \Sigma$, define the energy (or action functional):

$$\mathcal{E}(f) = \frac{1}{2}\int_{\Sigma} \|df\|^2 dV_{h_0}.$$

Above, we view the differential of f as a section $df \in \Gamma(T^*\Sigma \otimes f^*T\Sigma)$ and the norm $\|df\|^2$ (the *energy density*) is computed using the metrics h_0 and h, explicitly in local coordinates:

$$\|df\|^2 = h_0^{ij}\partial_i f^\alpha \partial_j f^\beta h_{\alpha\beta}(f).$$

Here, we employ Einstein summation convention with matching upper and lower indices being summed. Latin indices refer to domain variables and Greek indices to range variables; lastly,

$$f^\alpha = f \circ y^\alpha.$$

Definition 1.1. *A C^1-mapping $f : \Sigma \to \Sigma$ is harmonic if it is a critical point for the energy. Namely, given any C^1-variation through C^1-mappings $f_t : \Sigma \to \Sigma$, the first variation vanishes:*

$$\frac{d}{dt}\mathcal{E}(f_t)|_{t=0} = 0.$$

Throughout the rest of the notes, we will ignore all issues of regularity and assume all maps to be C^∞. This is not a major restriction for our purposes, the regularity theory for semi-linear elliptic PDE will guarantee the harmonic maps we meet are even real-analytic.

A smooth map $f : \Sigma \to \Sigma$ which is harmonic necessarily satisfies the Euler-Lagrange equations

$$\Delta_{h_0} f^\gamma + \Gamma^\gamma_{\alpha\beta}(f)\partial_i f^\alpha \partial_j f^\beta h_0^{ij} = 0 \tag{1.1}$$

for $\gamma = 1, 2$.

In these equations, Δ_{h_0} is the Laplace-Beltrami operator for the metric h_0 and the $\Gamma^\gamma_{\alpha\beta}$ are the Christoffel symbols for the target metric h. This is a semi-linear system of elliptic partial differential equations; if this sounds a bit arcane, for our purposes it assures that solutions to this equation have very strong regularity and uniqueness properties.

Observe that everything said above makes sense replacing Σ by a pair of closed Riemannian manifolds (M, g) and (N, h). Harmonic maps in this setting were formally introduced by Eells and Sampson [ES64] culminating in:

Theorem 1.2. *Let (M, g) and (N, h) be closed, smooth Riemannian manifolds such that (N, h) has negative sectional curvature. Then in every homotopy class of maps $[M \to N]$ there exists a smooth harmonic map $f : (M, g) \to (N, h)$. Furthermore, provided f does not map onto a closed geodesic in N, it is unique.*

Remark: The existence statement above is due to Eells-Sampson, while the uniqueness is due to Hartman [Har67].

In our setting, The above Theorem becomes:

Theorem 1.3. *There exists a unique harmonic map $f : (\Sigma, h_0) \to (\Sigma, h)$ homotopic (even isotopic) to the identity.*

In conformal coordinates, the Euler-Lagrange equations (1.1) take the form

$$\frac{\partial f}{\partial z \partial \overline{z}} + 2e^{-v}\frac{\partial}{\partial z}\left(e^{v(f)}\right)\frac{\partial f}{\partial z}\frac{\partial f}{\partial \overline{z}} = 0.$$

The form of this equation exposes a crucial invariance:

- In the case that the domain is a surface, the harmonic map depends only on the conformal class of the metric. This follows since the metric h_0 appears nowhere, only the conformal coordinate z.

For this reason, a harmonic map from a surface is linked to the holomorphic geometry of the surface. This is made precise by the following Proposition due to Hopf [Hop54],

Proposition 1.4. *Define a quadratic differential by,*

$$\phi := \phi(z)dz^2 = h\left(\frac{\partial f}{\partial z}, \frac{\partial f}{\partial z}\right)dz^2.$$

If f is harmonic, then ϕ is holomorphic.

Written in the conformal coordinates,

$$\phi(z) = e^{2v(f(z))}\frac{\partial f}{\partial z}\overline{\frac{\partial f}{\partial \overline{z}}}.$$

The ϕ associated to a harmonic map f is called the Hopf differential. This defines a map, which depends on a choice of conformal structure $\sigma \in \mathcal{T}$ in the Teichmüller space of isotopy classes of conformal structures on the surface:

$$\begin{aligned} N_\sigma : \mathcal{F} &\longrightarrow QD(\Sigma,\sigma) := H^0((\Sigma,\sigma),K^2) \\ h &\longmapsto \text{Hopf differential of unique harmonic map.} \end{aligned}$$

Here we make the distinction between the Tiechmüller space $\mathcal{T}$ and the Fricke space $\mathcal{F}$ of isotopy classes of hyperbolic metrics on Σ. The two are only identified after making the transcendental identification afforded by the Köebe-Poincaré uniformization Theorem.

The fundamental theorem which allows one to "do" Teichmüller theory with this approach is due to the efforts of many mathematicians, we mention Wolf [Wol89] and refer to the references therein:

Theorem 1.5. *For each $\sigma \in \mathcal{T}$, the map*

$$N_\sigma : \mathcal{F} \longrightarrow H^0((\Sigma,\sigma),K^2)$$

is continuous, injective and proper. Hence it is a homeomorphism since the latter space of sections is a vector space.

Remark: In fact, the mapping is real analytic (due to the fact that the harmonic map depends real analytically on parameters) for the real analytic structure on the Fricke space induced by its incarnation as a component of the character variety

$$\chi(\pi, \mathrm{PSL}\,(2,\mathbb{R})) = \mathrm{Hom}\,(\pi, \mathrm{PSL}(2,\mathbb{R}))\,/\mathrm{PSL}(2,\mathbb{R}).$$

The proof outlines as follows: the continuity follows from the well-posedness of solutions to the harmonic map equations, i.e. solutions depend continuously on the data. The injectivity follows from an easy application of the maximum principle, once an equation has been concocted to which it can be applied. The properness is the only piece requiring specific consideration and can be found in the paper of Wolf [Wol89]. Morally, it is proper because it takes more "energy" to stretch a hyperbolic surface onto one which far away from it in the Fricke space.

Remarkably, Hitchin [Hit87], Simpson [Sim92], Donaldson [Don87], and Corlette [Cor88] discovered that this is part (a base case!) of a profound correspondence:

{ X closed, Kähler manifold }$\longrightarrow$ { Reductive representations of $\pi_1(X)$ into a reductive Lie group } $\longrightarrow$ { Holomorphic objects over X }.

The particularly ingeniuous leap stems from the fact that in our above discussion of harmonic maps and Fricke-Teichmüller theory, we are missing part of the holomorphic objects over X alluded to in the diagram above. These holomorphic objects are named Higgs bundles. The Hopf differential is a remnant of the Higgs bundle which we will construct for any reductive representation $\rho : \pi \to \mathrm{SL}(n,\mathbb{C})$.

Before we tackle this, we must recall the basic calculus of vector bundles.

2. Calculus on Vector Bundles: metrics and connections on bundles

A thorough reference for the following material is Kobayashi's book [Kob87].

Let M be a smooth, closed manifold and $V \to M$ a smooth real or complex vector bundle over M. We denote the $C^\infty(M)$-module of smooth sections of V by $\Gamma(V)$.

Definition 2.1. *A connection (or covariant derivative) is a first order differential operator*

$$\nabla : \Gamma(V) \to \Gamma(T^*M \otimes V)$$

satisfying:

$$\nabla(s + s') = \nabla s + \nabla s'$$
$$\nabla f s = df \otimes \nabla s + f \nabla s$$

for all $s, s' \in \Gamma(V)$ and $f \in C^\infty(M)$.

Given two connections ∇, ∇', the difference satisfies

$$\nabla - \nabla'(fs) = f(\nabla - \nabla')s.$$

Hence, $\nabla - \nabla' \in \Omega^1(\mathrm{End}(V))$ where $\Omega^1(\mathrm{End}(V))$ is the space of one-forms on M with values in the endomorphisms of V. In slightly fancy language, the space of connections on V, denoted $\mathcal{A}(V)$, is an affine space with underlying vector space of translations $\Omega^1(\mathrm{End}(V))$.

Given $\nabla \in \mathcal{A}(V)$, there exists a skew-symmetric extension called the exterior covariant differential operator associated to ∇ :

$$d^\nabla : \Omega^k(V) \to \Omega^{k+1}(V)$$

where $\Omega^k(V)$ is the space of exterior differential k-forms with values in V. It is defined by enforcing the graded Leibniz rule: given $\omega \in \Omega^k(M)$ and $s \in \Gamma(V)$,

$$d^\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^k \omega \wedge \nabla s.$$

The introduction of this operator allows for a concise definition of the *curvature* of a connection,

Definition 2.2. *Given a connection $\nabla \in \mathcal{A}(V)$, the curvature 2-form is defined by*

$$F^\nabla := d^\nabla \circ \nabla \in \Omega^2(End(V)).$$

Over a trivializing open set $U \subset M$, a section may be written $s = s^i e_i$. The action of a connection ∇ is

$$\nabla s = (ds^j + s^i A_i^j) e_j,$$

where the connection coefficients A_i^j (relative to U) comprise a matrix of 1-forms on U. Then the curvature of ∇ acts via

$$F^\nabla(s) = s^i(dA_i^k + A_j^k \wedge A_i^j)e_k.$$

Now suppose M is a complex manifold and V is a smooth, complex vector bundle. The complexified co-tangent bundle of M splits into types

$$T^*M \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)}M \oplus T^{(0,1)}M.$$

Definition 2.3. *A pseudo-connection (or Cauchy-Riemann, or del-bar operator) is a first order differential operator*

$$\overline{\partial}_V : \Gamma(V) \to \Gamma(T^{(0,1)}M \otimes V)$$

such that

$$\overline{\partial}_V(s+s') = \overline{\partial}_V s + \overline{\partial}_V s'$$
$$\overline{\partial}_V(fs) = \overline{\partial} f \otimes s + f\overline{\partial}_V s.$$

This operator also has a graded skew-commutative extension to an operator which by abuse of notation we write the same way,

$$\overline{\partial}_V : \Omega^{p,q}(V) \to \Omega^{p,q+1}(V).$$

Similarly to connections, pseudo-connections are an affine space with underlying vector space of translations $\Omega^{0,1}(\mathrm{End}(V))$. The following integrability condition shows the importance of such operators.

Theorem 2.4. *Let $(V, \overline{\partial}_V)$ be a complex vector bundle with a pseudo-connection. Then V has a holomorphic structure whose holomorphic local sections are exactly those sections s satisfying $\overline{\partial}_V s = 0$ if and only if $\overline{\partial}_V^2 = 0$.*

Note that in the case that the manifold M is a surface, the condition of the above theorem is automatically satisfied since a Riemann surface carries no non-zero $(0,2)$ forms. In particular, given any connection ∇ on V, we may define a holomorphic structure via $\nabla^{0,1} = \overline{\partial}_V$.

Now we introduce metrics on vector bundles.

Definition 2.5. *An Hermitian metric on a complex vector bundle V is a smoothly varying family of hermitian forms:*

$$h : V_x \otimes V_x \to \mathbb{C}$$

on each fiber V_x over $x \in M$.

A connection $\nabla \in \mathcal{A}(V)$ is called *unitary* with respect to an Hermitian metric h if for all $s, s' \in \Gamma(V)$,

$$dh(s,s') = h(\nabla s, s') + h(s, \nabla s').$$

Supposing we have a holomorphic vector bundle $(V, \overline{\partial}_V)$ with an Hermitian metric,

Proposition 2.6. *There exists a unique unitary connection ∇ (called the Chern connection) such that $\nabla^{0,1} = \overline{\partial}_V$.*

From here forward, we will fix a Riemann surface structure $\sigma \in \mathcal{T}$ and denote the corresponding Riemann surface $X = (\Sigma, \sigma)$. Given a representation $\rho : \pi \to \mathrm{SL}(n, \mathbb{C})$, form the associated flat vector bundle

$$V_\rho := \widetilde{\Sigma} \times_\rho \mathbb{C}^n,$$

defined as a the quotient of $\widetilde{\Sigma} \times \mathbb{C}^n$ via the diagonal (left) action of π acting on the second factor via composition with the representation ρ. In a flat trivialization, the flat connection is simply the exterior differential acting component-wise on a (local) vector-valued function.

Next, consider a ρ-equivariant map

$$f : \widetilde{\Sigma} \to X_n,$$

where

$$X_n := \{A \in \mathrm{SL}(n, \mathbb{C}) \ |A = A^*, \ \det(A) = 1\}$$

is an explicit model for the symmetric space $\mathrm{SL}(n, \mathbb{C})/\mathrm{SU}(n)$. The left action of $\mathrm{SL}(n, \mathbb{C})$ on X_n is given by

$$g \cdot A = (g^{-1})^* A g^{-1}.$$

Note that such an equivariant map exists since it is equivalent to a section of the fiber bundle

$$\widetilde{\Sigma} \times_\rho X_n$$

which has contractible fibers. Let $\langle \ , \ \rangle$ denote the standard Hermitian inner product on $\mathbb{C}^n$. Given a pair of sections of $s, s' \in \Gamma(V_\rho)$, identify them with equivariant vector-valued maps $\widetilde{\Sigma} \to \mathbb{C}^n$. For $x \in \widetilde{\Sigma}$, f defines a pairing

$$H_f(s, s')(x) = \langle s(x), f(x)s'(x) \rangle.$$

If $\gamma \in \pi$ is acting by deck transformations,

$$\begin{aligned} H_f(s, s')(\gamma x) &= \langle s(\gamma x), f(\gamma x)s'(\gamma x) \rangle \\ &= \langle \rho(\gamma)s(x), (\rho(\gamma)^{-1})^* f(x)\rho(\gamma)^{-1}\rho(\gamma)s'(x) \rangle \\ &= \langle s(x), f(x)s'(x) \rangle \\ &= H_f(s, s')(x). \end{aligned}$$

Thus, H_f defines an Hermitian metric on V_ρ. Conversely, given a metric H on V_ρ, select a positively oriented unitary frame $\{e_i\}$ over the base point $x \in \widetilde{\Sigma}$. Parallel translation using the flat connection gives a global section of the unitary frame bundle which we will also denote $\{e_i\}$. Then define a ρ-equivariant map,

$$\begin{aligned} f : \widetilde{\Sigma} &\longrightarrow X_n \\ y &\longmapsto \{H(e_i(y), e_j(y))\}_{i,j=1,..,n}. \end{aligned}$$

The above two processes are inverse to one another.

Now, denote the flat connection on V_ρ by ∇. Given an Hermitian metric on V_ρ, split the connection (uniquely) as a unitary connection A plus an Hermitian endomorphism $\Psi \in \Omega^1(\mathrm{End}(V_\rho))$,

$$\nabla = A + \Psi.$$

The flatness of ∇ implies

$$0 = F^\nabla = F^A + d^A\Psi + \frac{1}{2}[\Psi, \Psi]. \tag{2.1}$$

The first term is the curvature of the unitary connection A which by unitarity is a 2-form with values in *skew-Hermitian* endomorphisms. The second term is a 2-form with values in Hermitian endomorphisms. Writing $\Psi = \alpha^i \otimes f_i$ for $\alpha^i \in \Omega^1(\Sigma)$ and $f_i \in \Gamma(\mathrm{End}(V_\rho))$,

$$d^A(\Psi)(s) = d\alpha^i \otimes f_i(s) - \alpha^i \wedge A(f_i)(s)$$

and $A(f_i)$ is defined in a trivialization via its action on sections $s = s^j e_j$,

$$\begin{aligned} A(f_i)(s^j e_j) &:= A(s^j f_{ij}^k e_k) \\ &= d(s^j f_{ij}^k) \otimes e_k + s^j f_{ij}^k A_k^l e_l. \end{aligned}$$

Lastly, the final term combines the wedge product on forms with the Lie bracket (commutator) on endormorphisms;

$$[\Psi, \Psi] := \alpha^i \wedge \alpha^j \otimes [f_i, f_j].$$

The commutator of two Hermitian endomorphisms is *skew-Hermitian*, thus decomposing (2.1) into Hermitian and skew-Hermitian pieces yields a pair of equations:

$$F^A + \frac{1}{2}[\Psi, \Psi] = 0, \tag{2.2}$$

$$d^A \Psi = 0. \tag{2.3}$$

Next, use the complex structure on $X = (\Sigma, \sigma)$ to decompose the covariant derivative d^A and Ψ according to type:

$$\begin{aligned} d^A &= \partial^A + \overline{\partial}^A, \\ \Psi &= \phi + \phi^{*_H}. \end{aligned}$$

Above, $\phi \in \Omega^{1,0}(\mathrm{End}(V_\rho))$, $\phi^{*_H} \in \Omega^{0,1}(\mathrm{End}(V_\rho))$ with the latter adjoint defined using the metric H and the type change given by sending dz to $d\overline{z}$. Since there are no $(0,2)$ nor $(2,0)$ forms on X, (2.2) and (2.3) simplify further,

$$\begin{aligned} F^A + [\phi, \phi^{*_H}] &= 0, \\ \overline{\partial}^A(\phi) + \partial^A(\phi^{*_H}) &= 0. \end{aligned} \tag{2.4}$$

By the discussion about holomorphic structures, the operator $\overline{\partial}^A$ induces a holomorphic structure on V_ρ as well as $\mathrm{End}(V_\rho)$. Wouldn't it be nice if $\overline{\partial}^A \phi = 0$, i.e. ϕ is holomorphic?

Definition 2.7. *The metric H is called harmonic if and only if $\overline{\partial}^A \phi = 0$.*

This definition is bolstered by the following crucial fact:

Proposition 2.8. *H is harmonic if and only if the associated equivariant map*

$$f : \widetilde{X} \to X_n$$

is harmonic.

Remark: Harmonicity is defined with respect to any conformal metric on the Riemann surface $\widetilde{X}$ and the left-invariant Riemannian metric on X_n. Note that we did not define harmonic for non-compact manifolds, nor for equivariant maps: the energy to be minimized here is:

$$\mathcal{E}(f) = \frac{1}{2} \int_D \|df\|^2 dV$$

where $D \subset \widetilde{X}$ is a fundamental domain for the action of π and dV is the volume element of our chosen conformal metric on X.

The analog of the Eells-Sampson theorem in the equivariant case is the following, first proved by Donaldson [Don87] in rank 2, then by Corlette [Cor88] in full generality (see also Labourie [Lab91]).

Theorem 2.9. *Let $\rho \in Hom(\pi, SL(n,\mathbb{C}))$. There exists a ρ-equivariant harmonic map*

$$f : \widetilde{X} \to X_n$$

if and only if the Zariski closure of the image of ρ is a reductive subgroup of $SL(n,\mathbb{C})$. Furthermore, the map is unique up to post-composition by an element which centralizes the image of ρ.

A reductive subgroup is one which acts completely reducibly, via the adjoint representation, on the Lie algebra $\mathfrak{sl}(n,\mathbb{C})$. The prototypical non-reductive subgroup is the subgroup of upper triangular matrices.

Let $\mathfrak{D}(X)$ denote the space of gauge isomorphism classes of flat vector bundles with harmonic metric (the *De-Rham* moduli space) and $\chi(\Sigma, \mathrm{SL}(n,\mathbb{C}))$ the character variety consisting of conjugacy classes of reductive representations $\pi \to \mathrm{SL}(n,\mathbb{C})$ (the *Betti* moduli space).

The above theorem yields a map, parameterized by a chosen point $\sigma \in \mathcal{T}$,

$$N_\sigma : \chi(\Sigma, \mathrm{SL}(n,\mathbb{C})) \to \mathfrak{D}(X).$$

The above map has an obvious inverse given by taking the holonomy of the flat connection which proves,

Theorem 2.10. *There is a family of isomorphisms $N_\sigma : \chi(\Sigma, SL(n,\mathbb{C})) \to \mathfrak{D}(X)$ parameterized by a point $\sigma \in \mathcal{T}$ where $X = (\Sigma, \sigma)$.*

Remark: One way to think about this isomorphism is as a section of the bundle over $\chi(\Sigma, \mathrm{SL}(n,\mathbb{C}))$ whose fiber over ρ consists of the (isomorphism classes of) Hermitian metrics on V_ρ. Any smooth section of this bundle yields an identification of the character variety with the space of reductive flat bundles equipped with the metric picked out by the chosen section. This leads to an interesting (albeit vague) question: is there another consistent choice of Hermitian metric on flat bundles which yields a geometrically rich deformation space?

3. Higgs bundles

Finally, we introduce the notion of a Higgs bundle.

Definition 3.1. *A rank-n Higgs bundle over X is a triple $\mathbb{V} = (V, \overline{\partial}_V, \phi)$ where $(V, \overline{\partial}_V)$ is a holomorphic vector bundle and $\phi \in H^0(X, K \otimes End(V))$.*

Remark: $\phi \in H^0(X, K \otimes \mathrm{End}(V))$ says exactly that ϕ is a holomorphic, endomorphism-valued 1-form on X. The tensor ϕ is called the *Higgs* field.

Definition 3.2. *A rank-n Higgs bundle $\mathbb{V}$ over a Riemann surface X is stable if and only if for every ϕ-invariant holomorphic sub-bundle $W \subset V$,*

$$\frac{deg(W)}{rk(W)} < \frac{deg(V)}{rk(V)}.$$

$\mathbb{V}$ is poly-stable if there exists stable Higgs bundles $\mathbb{W}_1, ..., \mathbb{W}_k$ such that

$$\mathbb{V} = \mathbb{W}_1 \oplus .... \oplus \mathbb{W}_k.$$

The critical Theorem linking Higgs bundles to the story which has unfolded thus far is due to Hitchin [Hit87] (in rank 2) and Simpson [Sim92] in general.

Theorem 3.3. *Let $\mathbb{V} = (V, \overline{\partial}_V, \phi)$ be a poly-stable Higgs bundle such that det(V) is the trivial holomorphic line bundle. Then there exists a unique (up to unitary automorphism) Hermitian metric H on V such that the Chern connection A of H satisfies*

$$F^A + [\phi, \phi^{*_H}] = 0 \tag{3.1}$$

Furthermore, if such a metric exists on any Higgs bundle $\mathbb{V}$, then $\mathbb{V}$ is poly-stable.

Let $\mathfrak{M}_{0,n}(X)$ be the moduli space of degree 0, rank n, poly-stable Higgs bundles with fixed trivial determinant (the *Doulbeaut* moduli space). Here, equivalence is defined up to holomorphic automorphisms commuting with the Higgs field.

Note that we have already seen (3.1) in line (2.4); (3.1) is satisfied if and only if the connection

$$A + \phi + \phi^{*_H}$$

is flat. Furthermore, Theorem 2.9 implies that the holonomy of this flat connection is reductive since the metric H is harmonic; namely $\overline{\partial}^A(\phi) = 0$. This defines a map,

$$\mathfrak{M}_{0,n}(X) \to \chi(\Sigma, \mathrm{SL}(n, \mathbb{C})).$$

Additionally, this map has an inverse since every reductive representation yields a flat bundle with a harmonic metric, which in turn gives rise to a Higgs bundle solving (3.1), thus a poly-stable Higgs bundle. Using suitable topologies (arising from the C^∞-topology on tensors and the topology of pointwise convergence on representations) on the Dolbeaut and Betti moduli space, the following theorem is called the *Non-Abelian Hodge correspondence:*

Theorem 3.4. *The map described above yields a homeomorphism:*

$$\mathfrak{M}_{0,n}(X) \simeq \chi(\Sigma, SL(n, \mathbb{C})).$$

Remark: It is very important to note that this homeomorphism:

(1) Depends on the point $\sigma \in \mathcal{T}$ in a complicated way.
(2) Passes through the transcendental procedure of constructing an equivariant harmonic map.

A very interesting (to the author) future direction is to explore what the (well developed) theory of harmonic maps into symmetric spaces might have to say about the geometric nature of this isomorphism. A basic, but difficult question is the following:

- Can one describe the space of all quasi-Fuchsian representations $\chi(\Sigma, \mathrm{SL}(2, \mathbb{C}))$ purely in terms of the associated Higgs bundles?

The space $\mathfrak{M}_{0,n}(X)$ has many fascinating structures (complex symplectic, hyper-Kähler, quasi-projective variety) which we will not explore here at all. We mention two very important features:

(1) Sending $\phi \to e^{i\theta}\phi$ not only preserves stability, but also preserves the harmonic metric. This action descends to the space $\mathfrak{M}_{0,n}(X)$ and has fixed points which are the critical sub-manifolds for a Morse-Bott function on $\mathfrak{M}_{0,n}(X)$. This allows one, in a number of cases, to compute the rational cohomology of $\mathfrak{M}_{0,n}(X)$ and in particular, the number of connected components. This is one of the greatest success stories involving the theory

of Higgs bundles, and is still an active area whose genesis lies in the epic paper of Atiyah and Bott [AB83].

(2) Let $(p_2, ..., p_n)$ be a basis of the conjugation invariant polynomials on $\mathrm{End}(\mathbb{C}^n)$ with $deg(p_i) = i$. Then given a Higgs field ϕ,

$$p_i(\phi) \in H^0(X, K^i)$$

and by the conjugation invariance this descends to a map:

$$F : \mathfrak{M}_{0,n}(X) \to \bigoplus_{i=2}^{n} H^0(X, K^i)$$

called the *Hitchin fibration.* This is a proper map whose generic fibers are Abelian varieties. With respect to a symplectic structure on $\mathfrak{M}_{0,n}(X)$, this is actually a moment map for an algebraically completely integrable Hamiltonian system. Namely, there exists a maximal set of independent Poisson commuting functions whose Hamiltonian vector fields generate flows which lie on the level sets of the Hitchin fibration. Lastly, the $H^0(X, K^2)$ entry is the Hopf differential of the harmonic map associated to that poly-stable Higgs bundle (exercise!).

4. Examples of Higgs bundles and construction of Hitchin component

We begin this section with the Higgs bundle "version" of the first section of these notes. Then we will close with Hitchin's construction of what is now known as the Hitchin component, a component of real representations of π into $\mathrm{SL}(n, \mathbb{R})$ naturally containing the Fricke space of hyperbolic uniformizations of Σ.

As before, we fix a point $\sigma \in \mathcal{T}$ and denote $X = (\Sigma, \sigma)$. Consider the short exact sequence of sheaves

$$1 \to \mathbb{Z}_2 \to \mathcal{O}^* \to \mathcal{O}^* \to 1,$$

where the second arrow takes any locally holomorphic non-vanishing function f to its square f^2. The relevant segment of the long exact sequence in sheaf cohomology is

$$H^1(X, \mathbb{Z}_2) \to H^1(X, \mathcal{O}^*) \to H^1(X, \mathcal{O}^*) \to H^2(X, \mathbb{Z}_2).$$

The arrow

$$w_2 : H^1(X, \mathcal{O}^*) \to H^2(X, \mathbb{Z}_2)$$

computes the mod 2-reduction of the degree (the second *Steifel-Whitney class*) of the line bundle represented by a class in $H^1(X, \mathcal{O}^*)$. Since the canonical bundle of holomorphic 1-forms K has even degree equal to $2g - 2$, it maps under w_2 to zero, hence, by the long exact sequence above, there exists a line bundle $K^{\frac{1}{2}}$ which squares to K. Furthermore, there are

$$|H^1(X, \mathbb{Z}_2)| = |\mathbb{Z}_2^{2g}| = 2^{2g}$$

such inequivalent choices; pick one. Form the holomorphic rank-2 vector bundle

$$V = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}.$$

Then,

$$K \otimes \mathrm{End}(V) = K \oplus K^2 \oplus \mathcal{O} \oplus K$$

whereby if $\alpha \in H^0(X, K^2)$,

$$\phi = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$$

is a well-defined Higgs field. Here, 1 is the constant function, a global section of the sheaf $\mathcal{O}$ of holomorphic functions on X.

This is clearly holomorphic as each entry is holomorphic. Furthermore, there are no non-zero $\phi-$invariant sub-bundles unless $\alpha = 0$ in which case the only invariant sub-bundle is $K^{-\frac{1}{2}}$ which is of negative degree. Hence, (V, ϕ) is a stable Higgs bundle. Additionally, $det(V) = \mathcal{O}$, thus Theorem 3.3 implies that there exists an Hermitian metric on H_α on V with Chern connection A_α such that

$$F^A = -[\phi, \phi^{*_H}]. \tag{4.1}$$

If the metric was not diagonal, then relative to the holomorphic splitting of V an off diagonal entry would appear in the connection form of A_α, but this would imply that the splitting was *not* holomorphic since the connection A is the Chern connection. Thus, the metric has the form

$$H_\alpha = \begin{pmatrix} h_\alpha^{-\frac{1}{2}} & 0 \\ 0 & h_\alpha^{\frac{1}{2}} \end{pmatrix}$$

where h_α is an Hermitian metric on the holomorphic tangent bundle K^{-1}. Now we compute,

$$\phi^{*_H} = H_\alpha^{-1} \overline{\phi}^T H_\alpha = \begin{pmatrix} 0 & h_\alpha \overline{1} \\ h_\alpha^{-1}\overline{\alpha} & 0 \end{pmatrix}.$$

Thus,

$$-[\phi, \phi^{*_H}] = \begin{pmatrix} 1 - h_\alpha^{-2}\alpha\overline{\alpha} & 0 \\ 0 & -1 + h_\alpha^{-2}\alpha\overline{\alpha} \end{pmatrix} h_\alpha dz \wedge d\overline{z}.$$

Note, $h_\alpha^{-2}\alpha\overline{\alpha} = \|\alpha\|^2_{h_\alpha}$ is a scalar-valued function; the norm of α with respect to h_α. The Chern connection takes the form

$$A_\alpha = \begin{pmatrix} \frac{1}{2}a_\alpha^- & 0 \\ 0 & \frac{1}{2}a_\alpha \end{pmatrix}$$

where a_α is the connection 1-form of the metric h_α on K^{-1}. Thus, (4.1) reduces to a single scalar equation:

$$F^{a_\alpha} = -2(1 - \|\alpha\|^2_{h_\alpha}) h_\alpha dz \wedge d\overline{z}.$$

Let's inspect what we have, when $\alpha = 0$ the above equation reads

$$F^{a_\alpha} = -2h_0 dz \wedge d\overline{z}.$$

This immediately implies that the real part of h_0 furnishes a metric of constant sectional curvature -4 on the surface Σ. Thus, this special case of solving the self-duality equations is equivalent to solving the uniformization theorem. Hitchin [Hit87] showed much more:

Theorem 4.1. *Consider the metric h_α above on K^{-1}. Then the expression,*

$$\hat{h}_\alpha = \alpha dz^2 + (1 + \|\alpha\|^2_{h_\alpha}) h_\alpha dz d\overline{z} + \overline{\alpha} d\overline{z}^2$$

defines a Riemann metric on K^{-1} which has sectional curvature equal to -4. This assignment gives a parameterization of the Fricke space by the space of holomorphic quadratic differentials.

Let us return briefly to the discussion at the beginning of these notes. The following facts are a rewarding exercise for the interested reader:

- The holomorphic quadratic differential α is the Hopf differential of the unique harmonic map isotopic to the identity between from $(\Sigma, \sigma) \to (\Sigma, \hat{h}_\alpha)$.
- The Hitchin map (choose the ad-invariant quadratic polynomial given by minus the determinant) $F : \mathfrak{M}_{0,2}(X) \to H^0(X, K^2)$ takes the Higgs field
$$\phi = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$$
to α. Thus, the Hitchin fibration admits a *section* whose image picks out an entire component of representations lying in the split real form of $\mathrm{SL}(2, \mathbb{C})$.

The moral of this story is that these examples of Higgs bundles are none other than the Harmonic maps parameterization of the Fricke space dressed is slightly fancier language (compare with [Wol89]). The strength of this language is that *it generalizes* and reveals a wealth of structure which was hidden before exploiting the holomorphic geometry.

5. The $\mathrm{SL}(n, \mathbb{R})$ Hitchin component

This section is a condensed presentation of the material in [Hit92].

We now arrive at the goal of these notes, the construction of the Hitchin component for the linear group $\mathrm{SL}(n, \mathbb{R})$. In hindsight, this is a natural generalization of the work in the previous section, and it shows the power of the Higgs bundle theory to reveal new objects, which have turned out to be very geometric.

There is a unique n-dimensional irreducible representation of $\mathrm{SL}(2, \mathbb{C})$ given by the action of $\mathrm{SL}(2, \mathbb{C})$ on homogeneous polynomials of degree $n-1$ in 2 variables. This representation is the $(n-1)$-th symmetric power of the standard 2-dimensional representation. Recall the vector bundle above,
$$V = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}.$$
The $(n-1)$-th symmetric power of this vector bundle is given by
$$S^{n-1}(V) = K^{\frac{n-1}{2}} \oplus K^{\frac{n-3}{2}} \oplus ... \oplus K^{-\frac{n-3}{2}} \oplus K^{-\frac{n-1}{2}}.$$
The Higgs field becomes
$$\phi = \begin{pmatrix} 0 & (n-1) & 0 & \cdots & & 0 \\ \alpha & 0 & 2(n-2) & \cdots & & 0 \\ 0 & \alpha & 0 & 3(n-3) & \cdots & 0 \\ \vdots & & & \ddots & \ddots & \vdots \\ 0 & & & & \ddots & (n-1) \\ 0 & 0 & \cdots & 0 & \alpha & 0 \end{pmatrix}.$$
Now, we have the liberty to do something with the Higgs fields: take $(n-1)$ elements
$$(\alpha_2, \alpha_3, ..., \alpha_n) \in \bigoplus_{i=2}^{n} H^0(X, K^i).$$

Form the new Higgs field, while keeping the holomorphic vector bundle $S^{n-1}(V)$ fixed,

$$\widetilde{\phi} = \begin{pmatrix} 0 & (n-1) & 0 & \cdots & & 0 \\ \alpha_2 & 0 & 2(n-2) & \cdots & & 0 \\ \alpha_3 & \alpha_2 & 0 & 3(n-3) & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \alpha_{n-1} & & & & \ddots & (n-1) \\ \alpha_n & \alpha_{n-1} & \cdots & \alpha_3 & \alpha_2 & 0 \end{pmatrix}.$$

A linear algebra calculation (a more clever argument can be found in Hitchin's paper [Hit92]) shows that the Higgs bundle $(S^{n-1}(V), \widetilde{\phi})$ is Higgs stable. Thus, the Theorem of Hitchin and Simpson (Theorem 3.3) guarantees the existence of an Hermitian metric H with Chern connection A such that the self-duality equations are satisfied.

The first question we wish to attack is the following: what are the properties of the holonomy of the flat connection

$$B = A + \widetilde{\phi} + \widetilde{\phi}^{*_H}?$$

For this we will use the following very important Proposition (see [Hit92], [Sim92]).

Proposition 5.1. *Let $\rho \in \chi(\Sigma, SL(n, \mathbb{C}))$ correspond to a Higgs bundle (V, ϕ). (Here the holomorphic structure on V is implicit). Then the Higgs bundle (V^*, ϕ^t) corresponds to the conjugate representation $\overline{\rho}$.*

Thus, fixed points (up to holomorphic automorphism conjugating the Higgs field) of the involution

$$\eta : (V, \phi) \mapsto (V^*, \phi^t)$$

correspond to *real* representations.

Returning to our Higgs bundle $(S^{n-1}(V), \widetilde{\phi})$, the anti-diagonal holomorphic automorphism

$$\begin{pmatrix} 0 & 0 & \cdots & & 1 \\ 0 & \cdots & & 1 & 0 \\ \vdots & & \iddots & & \vdots \\ 0 & 1 & \cdots & & 0 \\ 1 & 0 & \cdots & & 0 \end{pmatrix}$$

maps it to $(S^{n-1}(V)^*, \widetilde{\phi}^t)$. Hence, on the moduli space of Higgs bundles it is fixed by the involution η. Thus, by the previous Proposition 5.1 the holonomy of the flat connection B to which this Higgs bundle corresponds takes values in $\mathrm{SL}(n, \mathbb{R})$.

Next, we wish to show that we have constructed an entire component of real representations: the *Hitchin* component. There is a basis of conjugation invariant polynomials $\{p_2, ..., p_n\}$ such that

$$p_i(\widetilde{\phi}) = \alpha_i.$$

Using these to define the Hitchin fibration, we have constructed a *section*

$$s : \bigoplus_{i=2}^{n} H^0(X, K^i) \to \mathfrak{M}_{0,n}(X).$$

Simply by virtue of being a section, the map s is injective and has closed, connected image. An additional argument (see [Hit92]) shows that s takes values in the smooth part of the moduli space of Higgs bundles. Also, the differential of s is injective. Thus, using the implicit function theorem, the image of s is a closed, connected sub-manifold of $\mathfrak{M}_{0,n}(X)$. Using Corlette's theorem to identity Higgs bundles with reductive representations, we obtain a closed, connected sub-manifold of $\chi(\Sigma, \mathrm{SL}(n, \mathbb{R}))$.

At a smooth point of the character variety $\chi(\Sigma, \mathrm{SL}(n, \mathbb{R}))$, the index theorem can be used to show that its dimension is $|\chi(\Sigma)| \times \dim(\mathrm{SL}(n, \mathbb{R}))$. Meanwhile, a calculation employing the Riemann-Roch theorem yields that the dimension of the Hitchin base is equal to,

$$\begin{aligned} \dim\left(\bigoplus_{i=2}^{n} H^0(X, K^i)\right) &= |\chi(\Sigma)| \sum_{i=1}^{n-1}(2i+1) \\ &= |\chi(\Sigma)|(n^2-1) \\ &= |\chi(\Sigma)| \dim(\mathrm{SL}(n, \mathbb{R})). \end{aligned}$$

Remarkably, this is the same as the dimension of the character variety $\chi(\Sigma, \mathrm{SL}(n, \mathbb{R}))$! Thus, the section s which was previously known to be an immersion is a submersion as well. Applying the inverse function theorem, the image of s is open. As we already know it is closed and connected, this proves that the image of s is a single component of $\chi(\Sigma, \mathrm{SL}(n, \mathbb{R}))$. This is the component which Hitchin identified that is now known as the Hitchin component.

We close with a question: How can we infer geometric properties of the representations in the Hitchin component from differential-geometric properties of the equivariant harmonic map, or equivalently, from the harmonic metric on the associated flat bundle? For example, are there identifiable properties of the harmonic map which guarantee the representation is discrete?

At least for the author, these types of questions are some of the most fascinating surrounding the theory of Higgs bundles. Even in the rank 2 case, the situation is still very murky.

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GEOMETRIC CHARACTERIZATION OF HITCHIN REPRESENTATIONS

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Abstract. These notes are an attempt of summary of Labourie-Guichard's characterization of Hitchin representations established in [La] and [Gui$_3$] via the notion of hyperconvexity.

1. Hyperconvexity

We discuss the notions of hyperconvex representation and Anosov representations. In particular, we prove that hyperconvex representations are Anosov.

1.1. Hyperconvex representations.

Definition. A flag curve $\mathcal{F}\colon \partial_\infty \widetilde{S} \to \mathrm{Flag}(\mathbb{R}^n)$ is *hyperconvex* if it satisfies the following two conditions:

(1) for every k–tuple of distinct points $x_1, \ldots, x_k \in \partial_\infty \widetilde{S}$, for every k–tuple of integers $m_1, \ldots, m_k$ such that $m_1 + \cdots + m_k = n$,
$$\mathbb{R}^n = \mathcal{F}^{(m_1)}(x_1) \oplus \cdots \oplus \mathcal{F}^{(m_k)}(x_k);$$

(2) for every k–tuple of distinct points $x_1, \ldots, x_k \in \partial_\infty \widetilde{S}$, for every k–tuple of integers $m_1, \ldots, m_k$ such that $m_1 + \cdots + m_k = m \leq n$,
$$\lim_{\substack{(x_i)\to x \\ x_i \neq x}} \mathcal{F}^{(m_1)}(x_1) \oplus \cdots \oplus \mathcal{F}^{(m_k)}(x_k) = \mathcal{F}^{(m)}(x).$$

Note that a hyperconvex flag curve is in particular continuous. Besides, the image of the projective curve $\mathcal{F}^{(1)}\colon \partial_\infty \widetilde{S} \to \mathbb{RP}^{n-1}$ is a $\mathcal{C}^1$–embedded curve.

Definition. A homomorphism $\rho\colon \pi_1(S) \to \mathrm{PSL}_n(\mathbb{R})$ is a *hyperconvex representation* if there exists a ρ–equivariant, hyperconvex flag curve $\mathcal{F}_\rho\colon \partial_\infty \widetilde{S} \to \mathrm{Flag}(\mathbb{R}^n)$.

Example. Let $\rho_0\colon \pi_1(S) \to \mathrm{PSL}_n(\mathbb{R})$ be a *n–Fuchsian representation*, namely ρ_0 is a homomorphism of the form
$$\rho_0 = \iota \circ r$$
where: $r\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{R})$ is a Fuchsian homomorphism; and $\iota\colon \mathrm{PSL}_2(\mathbb{R}) \to \mathrm{PSL}_n(\mathbb{R})$ is the preferred homomorphism defined by the n–dimensional, irreducible representation of $\mathrm{SL}_2(\mathbb{R})$ into $\mathrm{SL}_n(\mathbb{R})$. Then ρ_0 is hyperconvex. Identify $\partial_\infty \widetilde{S}$ with $\mathbb{RP}^1$ viewed as the set of nonzero homogenous polynomials of the form $aX + bY$ up to scalar multiplication. The associated equivariant flag curve $\mathcal{F}_{\rho_0}$ is defined as follows: for every $i = 1, \ldots, n-1$, $\mathcal{F}_{\rho_0}^{(i)}\big([aX+bY]\big)$ is the i–dimensional subspace of homogenous polynomials of the form $a_0X^{n-1} + a_1X^{n-2}Y + \cdots + a_{n-1}Y^{n-1}$ that are multiples of $(aX+bY)^{n-i}$. One easily verifies that $\mathcal{F}_{\rho_0}$ is hyperconvex as it satisfies

both conditions (1) and (2). The projective curve $\mathcal{F}^{(1)}_{\rho_0}$ is called the *Veronese's embedding of* $\mathbb{RP}^1$ *in* $\mathbb{RP}^{n-1}$.

Exercise. Show that the "Veronese" flag curve is hyperconvex.

1.2. **Anosov representations.** Set $\bar{\mathbb{R}}^n = \mathbb{R}^n/\{\pm \mathrm{Id}\}$; note that $\mathrm{PSL}_n(\mathbb{R})$ acts on $\bar{\mathbb{R}}^n$. Given a homomorphism $\rho\colon \pi_1(S) \to \mathrm{PSL}_n(\mathbb{R})$, let $T^1S \times_\rho \bar{\mathbb{R}}^n \to T^1S$ be the flat twisted $\bar{\mathbb{R}}$–bundle associated with ρ.

Definition. A homomorphism $\rho\colon \pi_1(S) \to \mathrm{PSL}_n(\mathbb{R})$ is an *Anosov representation* if there exists a ρ–equivariant flag curve $\mathcal{F}_\rho\colon \partial_\infty \widetilde{S} \to \mathrm{Flag}(\mathbb{R}^n)$ that satisfies the following two conditions:

(1) for every pair of distinct points $x, y \in \partial_\infty \widetilde{S}$, for every $k = 1, \ldots, n-1$,
$$\mathbb{R}^n = \mathcal{F}^{(k)}_\rho(x) \oplus \mathcal{F}^{(n-k)}_\rho(y);$$

(2) let $(G_t)_{t\in\mathbb{R}}$ be the lift in the flat bundle $T^1S \times_\rho \bar{\mathbb{R}}^n$ of the geodesic flow $(g_t)_{t\in\mathbb{R}}$; the flag curve $\mathcal{F}_\rho$ provides a line splitting $V_1 \oplus \cdots \oplus V_n$ of $T^1S \times_\rho \mathbb{R}^n \to T^1S$ that is invariant under the action of $(G_t)_{t\in\mathbb{R}}$; we require the action of $(G_t)_{t\in\mathbb{R}}$ to be *Anosov* in the following sense; there exist a Riemannian metric $\| \ \|$ on the fibres of $T^1S \times_\rho \bar{\mathbb{R}}^n$ and some constants $A, a > 0$ such that, for every $i = 1, \ldots, n-1$, for every $t > 0$, for every $u \in T^1S$, for every unit vectors $X_i(u) \in V_i(u)$ and $X_{i+1}(u) \in V_{i+1}(u)$,
$$\frac{\|G_t X_i(u)\|_{g_t(u)}}{\|G_t X_{i+1}(u)\|_{g_t(u)}} \leq A e^{-at}.$$

A consequence of the Anosov dynamics is that an Anosov representation ρ admits a unique equivariant flag curve $\mathcal{F}_\rho$ satisfying (1) and (2). In addition, $\mathcal{F}_\rho\colon \partial_\infty \widetilde{S} \to \mathrm{Flag}(\mathbb{R}^n)$ is Hölder continuous. See [La, Gui$_3$, GuiW] for details.

Example. A n–Fuchsian $\rho_0 = \iota \circ r$ is Anosov. It is convenient to look at the situation in the universal cover $T^1\widetilde{S} \times \bar{\mathbb{R}}^n$. Given a base point $\widetilde{u}_0 \in T^1\widetilde{S}$, we can identify $T^1\widetilde{S}$ with $\mathrm{PSL}_2(\mathbb{R})$; the action of the geodesic flow $(g_t)_{t\in\mathbb{R}}$ on T^1S then identifies with the right action of the subgroup $\left\{ \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \right\}_{t\in\mathbb{R}}$ on $\mathrm{PSL}_2(\mathbb{R})$.

The Veronese flag curve $\mathcal{F}_{\rho_0}$ provides a line splitting $\widetilde{V}_1 \oplus \cdots \oplus \widetilde{V}_n$ of $T^1\widetilde{S} \times \bar{\mathbb{R}}^n$ that is invariant under the action of the lift $(G_t)_{t\in\mathbb{R}}$. Now, since T^1S is compact, to show that the flow $(G_t)_{t\in\mathbb{R}}$ is Anosov in the sense of (2), we may choose any suitable metric on the bundle $T^1S \times_{\rho_0} \bar{\mathbb{R}}^n$ for which the action is Anosov.

Pick a metric $\| \ \|_{\widetilde{u}_0}$ on the fibre $\bar{\mathbb{R}}^n_{\widetilde{u}_0}$ above $\widetilde{u}_0$; for every $\widetilde{u} = g\widetilde{u}_0 \in T^1\widetilde{S}$ where $g \in \mathrm{PSL}_2(\mathbb{R})$, for every $X \in \bar{\mathbb{R}}^n_{\widetilde{u}}$, set
$$\|X\|_{\widetilde{u}} := \left\| \iota(g)^{-1} X \right\|_{\widetilde{u}_0}.$$

This defines a metric on the fibres of the bundle $T^1\widetilde{S} \times \bar{\mathbb{R}}^n$. By construction, it is invariant under the left action of $\mathrm{PSL}_2(\mathbb{R})$ and thus descends to a well-defined metric on the bundle $T^1S \times_{\rho_0} \bar{\mathbb{R}}^n$. Observe that, if $\widetilde{u} = g\widetilde{u}_0$ where $g \in \mathrm{PSL}_2(\mathbb{R})$, then $g_t(\widetilde{u}) = (ga_tg^{-1})\widetilde{u}$ where $a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$. Because of the flat connection,

$$\begin{aligned} \|G_t X\|_{g_t(\widetilde{u})} &= \|X\|_{g_t(\widetilde{u})} \\ &= \|X\|_{(ga_tg^{-1})\widetilde{u}} \\ &= \left\| \iota(ga_tg^{-1})^{-1} X \right\|_{\widetilde{u}_0} \\ &= \left\| \iota(g)\iota(a_t)^{-1}\iota(g)^{-1})X \right\|_{\widetilde{u}_0} . \end{aligned}$$

It is easy to conclude from the above relation and the definition of the homomorphism $\iota\colon \mathrm{PSL}_2(\mathbb{R}) \to \mathrm{PSL}_n(\mathbb{R})$ that the action of the flow $(G_t)_{t\in\mathbb{R}}$ on the line splitting $\widetilde{V}_1 \oplus \cdots \oplus \widetilde{V}_n \to T^1S$ provided by the Veronese flag curve $\mathcal{F}_{\rho_0}$ satisfies the Anosov condition (2).

1.3. **Hyperconvex implies Anosov.** We now show the following result.

Theorem 1. *Let ρ be a hyperconvex representation. Then ρ is Anosov.*

Sketch of the proof. Consider the line bundle $V_{i+1}^* \otimes V_i \to T^1S$, endowed with the following metric $\|\ \|$ on the fibres: for every $u \in T^1S$, for every $\psi \in V_{i+1}^* \otimes V_i(u)$, set

$$\|\psi\|_u = \left| \frac{\langle \alpha_i, \psi(X_{i+1})\rangle \langle \alpha_{i+1}, Z_0\rangle}{\langle \alpha_{i+1}, X_{i+1}\rangle \langle \alpha_i, Z_0\rangle} \right|$$

where: $\widetilde{u} = (x_+, x_0, x_-) \in T^1\widetilde{S}$ is a lift of u; $Z_0 \in \mathcal{F}_\rho^{(1)}(x_0)$; $X_{i+1} \in \widetilde{V}_{i+1}(\widetilde{u})$; and α_j is a linear form with $\ker(\alpha_j) = \mathcal{F}_\rho^{(j-1)}(x_+) \oplus \mathcal{F}_\rho^{(n-j)}(x_-)$. Note that, because of the transversality condition (1) of a hyperconvex flag curve, $\|\ \|$ is well defined; moreover, it is clearly independent of the choices of Z_0, X_{i+1} and α_j.

To prove the condition (2), it is enough to show that the action of the flow $(G_t)_{t\in\mathbb{R}}$ on the line bundle $V_{i+1}^* \otimes V_i$ is exponentially contracting. To do so, we begin with proving that $\|G_t\psi\|_{g_t(u)}$ converges to 0 as t goes to $+\infty$.

Because of the flat connection,

$$\frac{\|G_t\psi\|_{g_t(u)}}{\|\psi\|_u} = \frac{\|\psi\|_{g_t(u)}}{\|\psi\|_u} = \left| \frac{\langle \alpha_{i+1}, Z_t\rangle \langle \alpha_i, Z_0\rangle}{\langle \alpha_i, Z_t\rangle \langle \alpha_{i+1}, Z_0\rangle} \right|$$

where: $g_t(\widetilde{u}) = (x_+, x_t, x_-)$, and $Z_t \in \mathcal{F}_\rho^{(1)}(x_t)$. Since $\lim_{t\to+\infty} \mathcal{F}_\rho^{(1)}(x_t) \oplus \mathcal{F}_\rho^{(i-1)}(x_+) = \mathcal{F}_\rho^{(i)}(x_+)$, we may assume that Z_t converges to a vector $Z_\infty \in \mathcal{F}_\rho^{(i)}(x_+) - \mathcal{F}_\rho^{(i-1)}(x_+)$. As a result,

$$\begin{aligned} &\lim_{t\to+\infty} \langle \alpha_i, Z_t\rangle = \langle \alpha_i, Z_\infty\rangle \neq 0; \\ &\lim_{t\to+\infty} \langle \alpha_{i+1}, Z_t\rangle = \langle \alpha_{i+1}, Z_\infty\rangle = 0. \end{aligned}$$

It follows that $\lim_{t\to+\infty} \|G_t\psi\|_{g_t(u)} = 0$.

The exponential contraction comes as a consequence of the compacity of T^1S. First of all, note that, since we are dealing with a flow, it is enough to show that there exists some $t_0 > 0$ so that, for every $t \geq t_0$, for every $u \in T^1S$, for every $\psi \in V_{i+1}^* \otimes V_i(u)$,

$$\|G_t\psi\|_{g_t(u)} < \frac{1}{2} \|\psi\|_u .$$

By contradiction, let $t_q \to +\infty$, $\widetilde{u}_q = (x_{+,q}, x_q, x_{-,q}) \in T^1\widetilde{S}$ and $\psi_q \in V_{i+1}^* \otimes V_i(u_q)$ be sequences for which $\left\|G_{t_q}\psi_q\right\|_{g_{t_q}(u_q)} \geq 1/2 \|\psi_q\|_{g_{t_q}(u_q)}$. By compacity of

T^1S and $\pi_1(S)$–invariance, we may assume that $\widetilde{u}_q$ converges to $\widetilde{u}_\infty = (x_+, x_\infty, x_-)$. Thus: set $g_{t_q}(\widetilde{u}_q) = (x'_{+,q}, x'_q, x'_{-,q})$; we have $x'_q \longrightarrow_{q\to+\infty} x_+$. Let

$$\frac{\left\|G_{t_q}\psi_q\right\|_{g_{t_q}(u_q)}}{\left\|\psi_q\right\|_{u_q}} = \frac{\left\|\psi_q\right\|_{g_{t_q}(u_q)}}{\left\|\psi_q\right\|_{u_q}} = \left|\frac{\left\langle \alpha_{i+1}, Z'_q\right\rangle \left\langle \alpha_i, Z_q\right\rangle}{\left\langle \alpha_i, Z'_q\right\rangle \left\langle \alpha_{i+1}, Z_q\right\rangle}\right| > \frac{1}{2}$$

where: $Z_q \in \mathcal{F}_\rho^{(1)}(x_q)$; and $Z'_q \in \mathcal{F}_\rho^{(1)}(x'_q)$. Since $\lim_{q\to+\infty} \mathcal{F}_\rho^{(1)}(x'_q) \oplus \mathcal{F}_\rho^{(i-1)}(x_+) = \mathcal{F}_\rho^{(i)}(x_+)$, we may assume that Z'_q converges to a vector $Z'_\infty \in \mathcal{F}_\rho^{(i)}(x_+) - \mathcal{F}_\rho^{(i-1)}(x_+)$, which, by the previous reasoning, yields a contradiction. See [Gui$_3$, §3], Proposition 18 for additional details. ☐

Note that in the above proof, we only make use of the transversality condition for a triple of flags, and of the limit condition (2) in the case where $i = 1$. Therefore, the hyperconvex condition appears to be notably more restrictive than Anosov.

2. Openess of hyperconvex representations

2.1. **Anosov 3–hyperconvex representations.**

Definition. A flag curve $\mathcal{F}\colon \partial_\infty\widetilde{S} \to \mathrm{Flag}(\mathbb{R}^n)$ is 3–*hyperconvex* if for every triple of distinct points x, y, $z \in \partial_\infty\widetilde{S}$, for every triple of integers j, k, l,

$$\mathbb{R}^n = \mathcal{F}^{(j)}(x) \oplus \mathcal{F}^{(k)}(y) \oplus \mathcal{F}^{(l)}(z).$$

Let $\mathcal{A}^3(S) \subset \mathcal{R}_{\mathrm{PSL}_n(\mathbb{R})}(S)$ be the set of Anosov representations that are 3–hyperconvex. Note that $\mathcal{A}^3(S)$ contains the set of hyperconvex representations $\mathcal{H}(S)$.

Lemma 2. *The set $\mathcal{A}^3(S)$ is open in $\mathcal{R}_{\mathrm{PSL}_n(\mathbb{R})}(S)$.*

Proof. Let ρ_0 be an Anosov representation. There exists an open subset $U \ni \rho_0$ such that the map

$$\begin{aligned} \Phi\colon U \times \partial_\infty\widetilde{S} &\to \mathrm{Flag}(\mathbb{R}^n) \\ (\rho, x) &\mapsto \mathcal{F}_\rho(x) \end{aligned}$$

is continuous.

Let $\partial_\infty\widetilde{S}^{3,*}$ be the set of triple of distinct points in $\partial_\infty\widetilde{S}$; recall that $\partial_\infty\widetilde{S}^{3,*}$ identifies with $T^1\widetilde{S}$. Consider the map

$$\begin{aligned} \Psi\colon U' \times \partial_\infty\widetilde{S}^{3,*} &\to \mathbb{N} \\ \big(\rho, (x,y,z)\big) &\mapsto \dim\big(\mathcal{F}_\rho(x) + \mathcal{F}_\rho(y) + \mathcal{F}_\rho(z)\big). \end{aligned}$$

By compacity of T^1S and $\pi_1(S)$–invariance of Ψ, it follows from the continuity of Φ that there exists $U' \subset U$ for which Ψ is constant and equal to n. See [Gui$_3$, §4], Proposition 20. ☐

The flag curve of an Anosov 3–hyperconvex representation satisfies the following regularity property.

Proposition 3. *Let $\mathcal{F}_\rho\colon \partial_\infty\widetilde{S} \to \mathrm{Flag}(\mathbb{R}^n)$ be the flag curve of an Anosov 3–hyperconvex representation ρ. Then, for every integers $m = k + l \le n$, for every $x \in \partial_\infty\widetilde{S}$,*

$$\lim_{\substack{(y,z)\to x \\ y\neq z}} \mathcal{F}_\rho^{(k)}(y) \oplus \mathcal{F}_\rho^{(l)}(z) = \mathcal{F}_\rho^{(m)}(x).$$

Note that the above limit implies the existence of a tangent line to the projective curve $\mathcal{F}_\rho^{(1)}$: the (image of the) curve $\mathcal{F}_\rho$ is $\mathcal{C}^1$. Besides, Theorem 1 shows that, if $\mathcal{F}_\rho$ is "smooth" enough, then ρ is Anosov. As it will become more apparent in the proof, Proposition 3 to some extent shows that the Anosov property is necessary to guarantee enough regularity for the flag curve $\mathcal{F}_\rho$.

Sketch of the proof in the case where $l = 1$. For $k \leq n$, consider the map

$$\eta^k \colon (y,z) \mapsto \begin{cases} \mathcal{F}_\rho^{(k-1)}(y) \oplus \mathcal{F}_\rho^{(1)}(z) & \text{if } y \neq z; \\ \mathcal{F}_\rho^{(k)}(y) & \text{if } y = z. \end{cases}$$

We want to show that η^k is continuous; clearly, we are only concerned with points of the form (y,y). Let $I \subset \partial_\infty \widetilde{S}$ be an interval; and let x_- be a point in the complement $\partial_\infty \widetilde{S} - I$. For every $y, z \in I$, set $\xi^{n-1}(y,z) = \eta^k(y,z) \oplus \mathcal{F}_\rho^{(n-k-1)}(x_-)$.

For every $x_+ \in I$, it follows (see exercise below) from the Anosov property for the line decomposition $V_1 \oplus \cdots \oplus V_n \to T^1 S$ and the 3–hyperconvexity that

$$\lim_{x \to x_+} \xi^{n-1}(x_+, x) = \xi^{n-1}(x_+, x_+).$$

Moreover, pick $x_0 \in I$. The contraction property enables us to show that the convergence is uniform, namely, for every $\varepsilon > 0$, for every $y^+ \in I$, there exist an interval $I' \subset I$ that contains y^+, and a constant $\delta > 0$, such that, for every $x_+, x \in I'$ with $\mathrm{dist}_{\partial_\infty \widetilde{S}}(x_+, x) \leq \delta$,

$$\mathrm{dist}_{\mathrm{Gr}^{n-1}(\mathbb{R}^n)}\big(\xi^{n-1}(x_+, x), \xi^{n-1}(x_+, x_+)\big) < \varepsilon.$$

Hence, by continuity of the map $y \mapsto \xi^{n-1}(y,y)$,

$$\lim_{(x_+, x) \to y_+} \xi^{n-1}(x_+, x) = \xi^{n-1}(y_+, y_+).$$

Finally, since $\eta^k = \xi^{n-1} \cap \eta^{k+1}$, we obtain the continuity of the map η^k for all $k \leq n-1$ by descending induction. The case when $l \geq 2$ is identical. See [Gui$_3$, §4], Proposition 20 for additional details. □

Exercise. Let $x_+ \in I$, and let $x_q \in I$ be a sequence converging to x_+ as $q \to \infty$. Using the techniques introduced in the proof of Theorem 1, show that, for every $\varepsilon > 0$, there exist an integer Q and a constant $C > 0$ such that, for every sequence of vectors $Z_q \in \mathcal{F}_\rho^{(k-1)}(x_+) \oplus \mathcal{F}_\rho^{(1)}(x_q) \oplus \mathcal{F}_\rho^{(n-k-1)}(x_-)$,

$$\mathrm{dist}_{\mathbb{R}^n}\big(Z_q, \mathcal{F}_\rho^{(k)}(x_+) \oplus \mathcal{F}_\rho^{(n-k-1)}(x_-)\big) \leq C \, \|Z_q\| \, \varepsilon.$$

Conclude that $\lim_{x \to x_+} \xi^{n-1}(x_+, x) = \xi^{n-1}(x_+, x_+)$.

2.2. **Openess.** Let $\mathcal{H}(S)$ be the set of hyperconvex representations.

Theorem 4. *$\mathcal{H}(S)$ is open in $\mathcal{R}_{\mathrm{PSL}_n(\mathbb{R})}(S)$.*

Let $\mathrm{Gr}^{i,+}(\mathbb{R}^n)$ be the Grassmannians of oriented vector spaces $\mathrm{Gr}^{i,+}(\mathbb{R}^n)$; it a 2–cover of $\mathrm{Gr}^i(\mathbb{R}^n)$. The observation below will come in very handy.

Lemma 5. *Let $\mathcal{F} \colon I \to \mathrm{Flag}(\mathbb{R}^n)$ be a hyperconvex flag curve defined on an (oriented) interval $I \subset \partial_\infty \widetilde{S}$. For every $i = 1, \ldots, n-1$, suppose that the map $\mathcal{F}^{(i)} \colon I \to \mathrm{Gr}^i(\mathbb{R}^n)$ lifts to $\mathcal{F}^{(i),+} \colon I \to \mathrm{Gr}^{i,+}(\mathbb{R}^n)$ so that*

$$\lim_{(y>z) \to x} \mathcal{F}^{(i-1),+}(y) \oplus \mathcal{F}^{(1),+}(z) = \mathcal{F}^{(i),+}(x).$$

Then there exists an orientation $\mathbb{R}^{n,+}$ *of* $\mathbb{R}^n$ *such that, for every* $n_1+\cdots+n_k=n$, *for every* $x_1>\cdots>x_k$,

$$\mathbb{R}^{n,+}=\mathcal{F}^{(n_1),+}(x_1)\oplus\cdots\oplus\mathcal{F}^{(n_k),+}(x_k).$$

Proof. Left to the reader as an exercise. □

Sketch of the proof of Theorem 4. The proof requires several steps. The key idea is the following: if $U\subset\mathcal{A}^3(S)$ is a contractible neighborhood of a hyperconvex representation ρ_0, then all representations in U are hyperconvex.

Let $\rho\in U$. By induction on k, we prove that, for every $x\in\partial_\infty\widetilde{S}$,

$$\mathrm{H}(k)\colon\begin{cases}\text{for every } k\text{–tuples of integers } m_1,\,\dots\,,\,m_k,\\[2ex] \lim_{\substack{(x_i)\to x\\ x_i\neq x}}\mathcal{F}_\rho^{(m_1)}(x_1)\oplus\cdots\oplus\mathcal{F}_\rho^{(m_k)}(x_k)=\mathcal{F}_\rho^{(m)}(x).\end{cases}$$

where $m=m_1+\cdots+m_k\leq n-1$.

Note that H(1) is true by continuity of the flag map $\mathcal{F}_\rho$, and H(2) is true by Proposition 3.

Let us focus attention on the special case of H(3) where $m_1=n-3$, $m_2=1$, $m_3=1$. We want to show that

$$\lim_{\substack{(x_i)\to x\\ x_i\neq x}}\mathcal{F}_\rho^{(n-3)}(x_1)\oplus\mathcal{F}_\rho^{(1)}(x_2)\oplus\mathcal{F}_\rho^{(1)}(x_3)=\mathcal{F}_\rho^{(n-1)}(x). \tag{1}$$

Let $I\subset\partial_\infty\widetilde{S}$ be an interval that contains x. Pick an orientation on I. It is enough to analyze the case where the above limit is taken under the extra condition that $x_1>x_2>x_3$. The advantage then resides in the possibility to lift the situation in the Grassmannians of oriented vector spaces $\mathrm{Gr}^{i,+}(\mathbb{R}^n)$. More precisely, let $\mathcal{F}_\rho^{(1),+}\colon I\to\mathrm{Gr}^{1,+}(\mathbb{R}^n)$ that lifts $\mathcal{F}_\rho^{(1)}$; for every $k\leq n-1$, define a lift $\mathcal{F}_\rho^{(k),+}$ and an orientation $\mathbb{R}^{n,+}$ for $\mathbb{R}^n$ via the following relations:

$$\mathcal{F}_\rho^{(k+1),+}(x)=\lim_{(y>z)\to x}\mathcal{F}_\rho^{(k),+}(y)\oplus\mathcal{F}_\rho^{(1),+}(z);$$
$$\mathbb{R}^{n,+}=\mathcal{F}_\rho^{(n-1),+}(y)\oplus\mathcal{F}_\rho^{(1),+}(z)\text{ if } y>z.$$

We begin with studying cluster points of converging sequences. Let $(x>)\ x_{1,q}>x_{2,q}>x_{3,q}$ be a triple of sequences that converge to $x\in\partial_\infty\widetilde{S}$. Assume that the sequence

$$P_q^{(n-1),+}=\mathcal{F}_\rho^{(n-3),+}(x_{1,q})\oplus\mathcal{F}_\rho^{(1),+}(x_{2,q})\oplus\mathcal{F}_\rho^{(1),+}(x_{3,q})$$

converges to the oriented hyperplan $P^{(n-1),+}\in\mathrm{Gr}^{n-1,+}(\mathbb{R}^n)$. By Proposition 3, $P^{(n-1)}$ contains $\mathcal{F}_\rho^{(n-2)}(x)$. Hence, for $w<x$, the sequence of oriented lines (3–hyperconvexity!)

$$Z_q^{(1),+}=P_q^{(n-1),+}\cap\mathcal{F}_\rho^{(2),+}(w)$$

converges to the oriented line $Z^{(1),+}=P^{(n-1),+}\cap\mathcal{F}_\rho^{(2),+}(w)$. We want to prove that

$$Z^{(1),+}=\mathcal{F}_\rho^{(n-1),+}(x)\cap\mathcal{F}_\rho^{(2),+}(w)$$

which will imply that $P^{(n-1),+}=\mathcal{F}_\rho^{(n-1),+}(x)$.

Let $D_q^{(n-1),+}$ and $E_q^{(n-1),+}$ be the sequences of oriented hyperplans defined by

$$D_q^{(n-1),+} = \mathcal{F}_\rho^{(n-2),+}(x_{1,q}) \oplus \mathcal{F}_\rho^{(1),+}(x_{3,q})$$
$$E_q^{(n-1),+} = \mathcal{F}_\rho^{(n-3),+}(x_{1,q}) \oplus \mathcal{F}_\rho^{(2),+}(x_{3,q}).$$

A first step is to show that both sequences converge to the oriented hyperplan $\mathcal{F}^{(n-1),+}(x)$. The convergence without the orientation nonsense is guaranteed by Proposition 3. Now, $\lim_{q\to+\infty} D_q^{(n-1),+} = \mathcal{F}^{(n-1),+}(x)$ is true by definition of the lift $\mathcal{F}^{(n-1),+}$. For the sequence $E_q^{(n-1),+}$, it is enough to show that

$$E_q^{(n-1),+} \oplus \mathcal{F}_\rho^{(1),+}(w) = \mathcal{F}_\rho^{(n-1),+}(x) \oplus \mathcal{F}_\rho^{(1),+}(w)(= \mathbb{R}^{n,+}).$$

This is the moment when the orientation nonsense plays a central rôle. The open subset U being contractile, we can deform the flag curve $\mathcal{F}_\rho$ into the hyperconvex flag curve $\mathcal{F}_{\rho_0}$ through a continuous family of Anosov 3–hyperconvex curves $(\mathcal{F}_{\rho_t})_{t\in[0,1]} \subset U$ with $\mathcal{F}_{\rho_1} := \mathcal{F}_\rho$. Observe then that the lift $\mathcal{F}_{\rho,+}$ for $\mathcal{F}_\rho$ extends a lift $(\mathcal{F}_{\rho_t,+})_{t\in[0,1]}$ so that, for every $t \in [0,1]$,

$$\mathcal{F}_{\rho_t}^{(n-3),+}(x_{1,q}) \oplus \mathcal{F}_{\rho_t}^{(2),+}(x_{3,q}) \oplus \mathcal{F}_{\rho_t}^{(1),+}(w)$$

defines the same orientation on $\mathbb{R}^n$. When $t = 0$, the Lemma 5 applies to the hyperconvex flag curve $\mathcal{F}_{\rho_0}$: it guarantees that the orientations "behave well" when we take limits. As a result,

$$\begin{aligned}
E_q^{(n-1),+} \oplus \mathcal{F}_\rho^{(1),+}(w) &= \mathcal{F}_{\rho_1}^{(n-3),+}(x_{1,q}) \oplus \mathcal{F}_{\rho_1}^{(2),+}(x_{3,q}) \oplus \mathcal{F}_{\rho_1}^{(1),+}(w)\\
&= \mathcal{F}_{\rho_0}^{(n-3),+}(x_{1,q}) \oplus \mathcal{F}_{\rho_0}^{(2),+}(x_{3,q}) \oplus \mathcal{F}_{\rho_0}^{(1),+}(w)\\
&= \mathcal{F}_{\rho_0}^{(n-1),+}(x) \oplus \mathcal{F}_{\rho_0}^{(1),+}(w)\\
&= \mathcal{F}_{\rho_1}^{(n-1),+}(x) \oplus \mathcal{F}_{\rho_1}^{(1),+}(w).
\end{aligned}$$

We can conclude via the following observation. Consider the two sequences

$$Z^{(1),+} \oplus D_q^{(n-1),+}$$
$$Z^{(1),+} \oplus E_q^{(n-1),+}$$

of $\mathrm{Gr}^{n,+}(\mathbb{R}^n)$. It is easy to see that these sequences have opposite orientations. Moreover, recall that the limits of both sequences each contain $Z^{(1),+}$ and $\mathcal{F}_\rho^{(n-1),+}(x)$. Therefore, $Z^{(1)}$ must be contained in $\mathcal{F}_\rho^{(n-1)}(x)$.

Regarding the existence of the limit (1) (un peu passé sous silence..): it comes as a consequence that (nonoriented and oriented) Grassmannians are compact spaces. Therefore, every sequence that admits a unique cluster point is convergent. See [Gui$_3$, §5], Theorem 23 for additional details. □

3. Closedness of hyperconvex representations

Theorem 6. *The set of hyperconvex representations $\mathcal{H}(S)$ is equal to the union of the Hitchin components of $\mathcal{R}_{\mathrm{PSL}_n(\mathbb{R})}(S)$.*

The proof of Theorem 6 strongly relies on algebraic group techniques; in other words, on aime ou on n'aime pas... It makes great use of results established in [Gui$_2$].

Here are two results that are essential in the following; the first one is a rather easy observation, while the second one requires hard work and is one of the main keys in the proof of Theorem 6.

Lemma 7. *Let ρ be a hyperconvex representation. Then ρ is strongly irreducible.*

Proof. Left to the reader as an exercise. See [Gui_3, §3], Propositions 14 and 15. □

Lemma 8. *Let ρ be a representation that is strongly irreducible and limit of hyperconvex representations. Then ρ is hyperconvex.*

Proof. Left to the reader as a nightmare. See [La, §9]. □

3.1. **The case where the genus is "large".** Let us denote by $\overline{\mathcal{H}(S)}$ be the adherence of the set of hyperconvex representations $\mathcal{H}(S)$. If the genus of the surface is large enough (in relation to n), Theorem 6 comes as a consequence of the following proposition, that, along with Lemma 8, constitute the keys of the proof.

Proposition 9. $\overline{\mathcal{H}(S)}$ *is open in* $\mathcal{R}_{\mathrm{PSL}_n(\mathbb{R})}(S)$.

Proof. See [Gui_3, §6], Proposition 26. □

As a consequence of Proposition 9, since $\mathcal{H}(S)$ contains the n–Fuchsian representations, $\overline{\mathcal{H}(S)}$ contains the Hitchin components.

Sketch of the proof of Theorem 6. By Proposition 9, $\overline{\mathcal{H}(S)}$ is the union of connected components of $\mathcal{R}_{\mathrm{PSL}_n(\mathbb{R})}(S)$; in particular, since $\mathcal{H}(S)$ contains all n–Fuchsian representations, $\overline{\mathcal{H}(S)}$ contains the Hitchin components. Besides, it is a consequence of [Hit] that all representations in the Hitchin components are strongly irreducible; hence, by Lemma 8,

$$\overline{\mathcal{H}(S)} \cap \text{Hitchin components} = \mathcal{H}(S) \cap \text{Hitchin components}$$

Finally, again due to [Hit], representations that are not Hitchin can be deformed into representations valued in the compact Lie group $\mathrm{PSO}_n(\mathbb{R})$; as a result, such components can not contain any discrete representation. □

3.2. **The general case.** The key lies in two simple observations that enable us to go from the large genus case back to the small genus case.

Lemma 10. *Let Γ be a finite index subgroup of $\pi_1(S)$. Then*

(1) $\rho\colon \pi_1(S) \to \mathrm{PSL}_n(\mathbb{R})$ *is hyperconvex if and only if* $\rho\colon \Gamma \to \mathrm{PSL}_n(\mathbb{R})$ *is hyperconvex;*
(2) $\rho\colon \pi_1(S) \to \mathrm{PSL}_n(\mathbb{R})$ *is Hitchin if and only if* $\rho\colon \Gamma \to \mathrm{PSL}_n(\mathbb{R})$ *is Hitchin.*

Proof. Left to the reader as an exercise. □

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A PRIMER ON COHOMOLOGICAL METHODS IN REPRESENTATION THEORY OF SURFACE GROUPS

TOBIAS HARTNICK

1. Introduction

These are extended notes from a series of talks that the author delivered in a workshop on Higher Teichmüller Thurston Theory in Northport, Maine in July 2013. The main story covered in this series of talks is centered around the following three aspects:

- We want to introduce some *basic cohomological tools*, which are useful in general representation theory of groups (and beyond). In particular, we discuss Gromov-Traubel-Johnson's *bounded cohomology* of discrete groups.
- We give one particularly simple (but in some sense prototypical) example which illustrates how cohomological tools can be used to study representations of surface groups. Namely, we give a purely cohomological condition (taken from [1]), which ensures that a given representation of a surface group into a Lie group is *faithful* and has *discrete image*.
- We explain how various geometric and/or topological assumptions (maximal Toledo invariant, existence of an Anosov structure over the Shilov boundary) imply that a representation is discrete and faithful by *exploiting this cohomological criterion.* Thereby we prove some simple special cases of results from [8] and [1].

We emphasize that there are many other things about representations besides discreteness and faithfulness that one can study using cohomological tools, even for surface groups; however, for limitations of time and space we will mostly focus on these two properties.

We have decided to present the material in a systematic rather than in a historical order. Let us point out right from the beginning that none of the results presented here are new, albeit a few details are presented differently than in the standard literature. All the material concerning group cohomology presented here is completely standard and can be found in any textbook on the subject, see e.g. [2]. Since it is easily accessible, we give basically no proofs. The modern theory of bounded group cohomology is not yet completely accessible in textbook form (although [12] covers some major parts of it). The two dissertations [29, 3] provide a good overview. Classical sources include [27, 23, 26, 22].

The whole uncensored story how bounded cohomology can be used to study representations of surface group is told in the recent survey [5], which covers in particular the breakthrough results from [8]. The present note was written with an eye towards that survey. This means that almost everything we say here, is also said in

[5] in greater generality. In particular, while we restrict ourselves to closed surfaces, [5] treats also the case of surfaces with boundary. It is our hope that the present notes can serve as an elementary introduction to [5]. Concerning bibliographical references we strongly recommend the reader to take a look at [5], since we will not be able to discuss or even repeat the more than 100 references here. It is not quite clear to the author what the historical starting point of this whole circle of ideas actually is, but Milnor's famous paper [28] was certainly a major early inspiration.

Let us mention explicitly the few places in which we deviate from [5]. Firstly and most importantly, in proving discreteness and faithfulness of maximal representations we make us of a very general criterion, which was established only recently in [1]. This allows us to reprove at the same time discreteness and faithfulness of certain Anosov representations. The corresponding part of these notes is taken almost literally from [1]. Our second deviation concerns the treatment of the bounded Euler class, which is one of the major tools in the cohomological study of surface groups. A famous theorem of Ghys says that it is a complete quasi-conjugacy invariant of representations, and this theorem can be invoked to detect trivial subgroups of surface groups using bounded cohomology. However, what is actually needed in this argument is only a very special case of Ghys' theorem, and we present here an elementary proof of this special case due to Michelle Bucher and the author. For the reader interested in the full story we recommend Ghys' original papers [19, 20]. Our third deviation from the standard literature concerns our treatment of the second bounded cohomology. We show that second bounded cohomology classes can be identified with equivalence classes of certain objects that we call *quasi-corners*, and which generalize central extensions and quasimorphisms at the same time. We find this language very useful from a pedagogical point of view and believe that it helps to clarify some statements, but it is essentially just a reformulation of very old ideas (going all the way back to Poincaré).

Finally, some remarks concerning prerequisites. These talks were given in front of a very specific audience, which was familiar with representations of surface groups in general and Anosov representations in particular, but not necessarily with group cohomology. We therefore develop all cohomological tools from scratch (some without proof), while we take various basic facts in hyperbolic geometry for granted. Probably our greatest omission is that we do not discuss the definition of Anosov representations at all. However, the reader who wants to close this gap finds more than sufficient information in the contributions of Spencer Dowdall and Tengren Zhang in this collection [17, 31]. The odd sections of the body of this article (i.e. 3, 5 and 7) do not refer to surface groups or representations in any essential way (except for examples), and provide an introduction to bounded cohomology which might be of independent interest. However, we warn the reader that this introduction is heavily biased by the applications we have in mind. In particular, we focus completely on (bounded) cohomology in degree 2.

Throughout these notes we restrict attention to closed oriented surfaces. This is morally wrong, since the true power of the bounded cohomology machinery only becomes visible in the case of surfaces with boundary. However, since our main goal is to be elementary this restriction was, unfortunately, unavoidable. We encourage the reader to read the uncensored story in [5].

Contents

2. Reminder of group cohomology

In this section we recall some basic facts from group cohomology. We refer to the textbook literature for proofs. What we actually need is rather little, so any textbook on group cohomology such as [2] will do.

2.1. **What is group cohomology?** In textbooks on homological algebra, group cohomology is often introduced as follows:

Definition 2.1. Let G be a group, R a ring and $RG - \mathfrak{Mod}$ the category of left RG-modules. Let

$$(-)^G : RG - \mathfrak{Mod} \to R - \mathfrak{Mod}, \quad V \mapsto V^G := \{v \in V \mid g.v = v\}$$

be the functor of G-invariants. Denote by

$$H^n(G; -) := \mathcal{D}^n(-)^G : RG - \mathfrak{Mod} \to \mathfrak{Mod}$$

its n-th derived functor. Then $H^n(G; V)$ is called the n-th group cohomology of G with coefficients in V.

What does this mean?

- Firstly, for every RG-module V there is a certain R-module $H^n(G; V)$, and the assignment $V \mapsto H^n(G; V)$ is functorial.
- In order to compute the isomorphism class of $H^n(G; V)$ one proceeds as follows: One first chooses an augmented co-resolution
$$0 \to V \to I^0 \to I^1 \to I^2 \to \dots$$
of V by injective RG-modules. One then deletes the augmentation and applies the functor $(-)^G$ to obtain a co-complex
$$0 \to (I^0)^G \to (I^1)^G \to (I^2)^G \to \dots$$
The cohomology of this cocomplex is then precisely $H^\bullet(G; V)$.

If these words don't mean anything to you, here is what you should remember: The invariants $H^\bullet(G; V)$, which depend only on G and V can be computed from many different resolutions. Therefore the same cohomology class can take very different meanings (geometric/topological/algebraic) in different contexts. This is the reason why group cohomology can be used to translate algebraic into geometric problems and vice versa. Fortunately, we will work with only a few very specific resolutions, which we discuss in detail in the sequel. We will be mostly interested in the case where $R \in \{\mathbb{Z}, \mathbb{R}\}$ and $V = R$ is the trivial RG-module.

2.2. **The topological resolution.** The following lemma links the purely algebraic definition of group cohomology to topology. Various version of this result were discovered independently by the American school around Eilenberg and McLane and the Swiss school around Eckmann and Hopf in the 1940s. Historically, this was the starting point of the modern theory of group cohomology.

Lemma 2.2. (i) *For every group G there exists a space BG, unique up to homotopy, such that $\pi_1(BG) = G$ and the universal covering EG of BG is contractible.*

(ii) $\mathbb{Z} \to C^0_{\text{sing}}(EG) \to C^1_{\text{sing}}(EG) \to \dots$ *is an augmented resolution of $\mathbb{Z}$.*

(iii) $H^\bullet(G; \mathbb{Z}) \cong H^\bullet(BG; \mathbb{Z})$.

Example 2.3. (i) $B\mathbb{Z} = \mathbb{S}^1$, hence

$$H^\bullet(\mathbb{Z}; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}.$$

(This is our short-hand notation to mean that $H^0(\mathbb{Z}; \mathbb{Z}) = \mathbb{Z}$, $H^1(\mathbb{Z}; \mathbb{Z}) = \mathbb{Z}$ and $H^n(\mathbb{Z}; \mathbb{Z}) = 0$ for $n > 1$.)

(ii) If $\Gamma_g := \langle a_1, b_1, \dots, a_g, b_g \mid \prod_{j=1}^g [a_j, b_j] \rangle$, then $B\Gamma_g = \Sigma_g$, the closed surface of genus g. Thus

$$H^\bullet(\Gamma_g; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}^{2g} \oplus \mathbb{Z}.$$

Historically, group cohomology was first *defined* as the cohomology of the classifying space. The usefulness of the derived functor interpretation only became apparent later on.

2.3. **The (in-)homogeneous bar resolution.** The homogeneous bar resolution is a combinatorial model for cohomology based on the injective resolution with

$$I^n := C^n(G; V) := \mathrm{Map}(G^{n+1}, V)$$

and *homogeneous differentials*

$$d : C^{n-1}(G; V) \to C^n(G; V), \quad dc(g_0, \dots, g_n) = \sum_{j=0}^n (-1)^j c(g_0, \dots, \widehat{g_j}, \dots, g_n).$$

Thus,

$$H^\bullet(G; V) = H^\bullet(C^0(G; V)^G \to C^1(G; V)^G \to C^2(G; V)^G \to C^3(G; V)^G \to \dots)$$

Note that there are isomorphisms

$$\iota_n : C^n(G; V)^G := \mathrm{Map}(G^{n+1}, V)^G \cong \mathrm{Map}(G^n; V)$$

given explicitly by

$$(\iota_n f)(g_1, \dots, g_n) = f(e, g_1, g_1 g_2, \dots, g_1 \cdots g_n), \quad (\iota_n^{-1} h)(g_0, \dots, g_n) = g_0.h(g_0^{-1} g_1, g_1^{-1} g_2, \dots, g_{n-1}^{-1} g_n).$$

Thus, if we define $\partial^n := \iota_n^{-1} \circ d \circ \iota_n$, then

$$H^\bullet(G; V) = H^\bullet(\mathrm{Map}(G^0, V) \xrightarrow{\partial^0} \mathrm{Map}(G^1, V) \xrightarrow{\partial^1} \mathrm{Map}(G^2, V) \xrightarrow{\partial^2} \mathrm{Map}(G^3, V) \to \dots).$$

This is called the *inhomogeneous bar resolution.*

Exercise 2.4. (i) Give explicit formulas for $\partial^0, \partial^1, \partial^2$.

(ii) Use the inhomogeneous bar resolution to show that

$$H^0(G;V) \cong V^G,$$

and if $V = R$ is the trivial RG-module then

$$H^1(G;R) \cong \mathrm{Hom}(G,R).$$

2.4. **The second cohomology.** According to Exercise 2.4, a 2-cocycle for the inhomogeneous bar resolution of the trivial module R is a function $c : G^2 \to R$ satisfying

$$\partial^2 c(g,h,k) = c(h,k) - c(gh,k) + c(g,hk) - c(g,h) = 0.$$

A 2-coboundary is a function $c : G^2 \to R$ for which there exists $f : G \to R$ such that

$$c(g,h) = \partial^1 f(g,h) = f(h) - f(gh) + f(h).$$

What is the algebraic meaning of a cohomology class $[c] \in H^2(G;R)$?

Definition 2.5. A *central extension* of G by R is a short exact sequence of the form

$$\xi = (0 \to R \xrightarrow{i} \widetilde{G} \xrightarrow{p} G \to 1) \tag{2.1}$$

with $R < C(\widetilde{G})$, the center of $\widetilde{G}$. A *right section* σ of the central extension (2.1) is a map $\sigma : G \to \widetilde{G}$ with $p\sigma = 1_G$. Two central extensions are *equivalent* if there exists an isomorphism making the diagram

$$\begin{tikzcd}
& & \widetilde{G}_1 \arrow[dd, "\cong"] \arrow[dr] & & \\
0 \arrow[r] & R \arrow[ur] \arrow[dr] & & G \arrow[r] & 1 \\
& & \widetilde{G}_2 \arrow[ur] & &
\end{tikzcd}$$

commute.

Proposition 2.6 (Classification of central extensions)**.** (i) *Let a central extension ξ as in* (2.1) *be given and let σ be a right-section. Then the map*

$$c_\sigma(g,h) := i^{-1}(\sigma(g)^{-1}\sigma(gh)\sigma(h)^{-1})$$

is a cocycle.

(ii) *If ρ is another right-section, then the difference $c_\sigma - c_\rho$ is a coboundary. Thus the cohomology class $e(\xi) := [c_\sigma] \in H^2(G;R)$ depends only on ξ.*

(iii) *The map $\xi \mapsto e(\xi)$ is an isomorphism between the set* $\mathrm{Ext}(G;R)$ *of isomorphism classes of central extensions of G by R and $H^2(G;R)$.*

In the sequel we refer to $e(\xi)$ as the *Euler class* of the central extension ξ.

2.5. **Functoriality and the lifting obstruction.** An important property of the cohomology functors $H^\bullet(H;R)$ with trivial coefficients is that they are also (contravariantly) functorial in the group variable. Explicitly, if $\rho : H \to G$ is a homomorphism and $c \in \mathrm{Map}(G^n;R)$ is an inhomogeneous then

$$\rho^* c(h_1, \dots, h_n) := c(\rho(h_1), \dots, \rho(h_n))$$

defines an inhomogeneous H-cocycle, and we obtain a map

$$\rho^* : H^\bullet(G;R) \to H^\bullet(H;R), \quad [c] \mapsto [\rho^* c]$$

in the opposite(!) direction. This map can also be defined more abstractly; either way, one can show that it is compatible with the geometric resolution: If H and G are groups and $\rho : H \to G$ is a homomorphism, then there exists a ρ-equivariant map $\varphi : BH \to BG$ (unique up to homotopy) and the diagram

$$\begin{tikzcd}
H^\bullet(G;R) \arrow[r, "\rho^*"] \arrow[d, "\cong"'] & H^\bullet(H;R) \arrow[d, "\cong"] \\
H^\bullet(BG;R) \arrow[r, "\varphi^*"] & H^\bullet(BH;R)
\end{tikzcd}$$

commutes.

We now specialize again to degree 2: Let $e \in H^2(G;\mathbb{R})$ and $\rho : H \to G$ be a homomorphism. What is the meaning of $\rho^* e$? Recall that, by Proposition 2.6 there exists a central extension ξ (unique up to isomorphism) such that $e = e(\xi)$. Assume that ξ is given as in (2.1). We say that the homomorphism ρ *lifts* over the central extension ξ if there exists a homomorphism $\widetilde{\rho} : H \to \widetilde{G}$ making the diagram

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r] & \widetilde{G} \arrow[r] & G \arrow[r] & 1 \\
 & & & H \arrow[ul, dotted, "\widetilde{\rho}"] \arrow[u, "\rho"'] &
\end{tikzcd}$$

commute.

Proposition 2.7 (Lifting obstruction)**.** *Let $e = e(\xi) \in H^2(G;R)$ and $\rho : H \to G$ be a homomorphism. Then $\rho^* e = 0 \in H^2(H;R)$ if and only if ρ lifts over the extension ξ. Moreover, if $\rho^* c_\sigma = du$ for some $u : H \to R$, then an explicit lift $\widetilde{\rho}$ is given by*

$$\widetilde{\rho}(h) = \sigma(\rho(h)) \cdot i(u(h)). \tag{2.2}$$

While the abstract obstruction result is standard, the explicit formula for the lift is not always given in the literature. Since we will heavily use this formula later on, we provide a proof:

Proof. A simple computation shows that if $\rho^* c_\sigma = du$ and $\widetilde{\rho}$ is defined as in (2.2) then

$$\begin{aligned}
\widetilde{\rho}(\gamma_1\gamma_2) &= \sigma(\rho(\gamma_1\gamma_2)) i(u(\gamma_1\gamma_2)) = \sigma(\rho(\gamma_1)\rho(\gamma_2)) \cdot i(u(\gamma_1\gamma_2)) \\
&= \sigma(\rho(\gamma_1)) i(e_\sigma(\rho(\gamma_1), \rho(\gamma_2)))\sigma(\rho(\gamma_2)) i(u(\gamma_1\gamma_2)) \\
&= \sigma(\rho(\gamma_1)) i(du(\gamma_1, \gamma_2))\sigma(\rho(\gamma_2)) i(-u(\gamma_1\gamma_2))
\end{aligned}$$

$$\begin{aligned} &= \sigma(\rho(\gamma_1))i(u(\gamma_2))i(-u(\gamma_1\gamma_2))i(u(\gamma_1))\sigma(\rho(\gamma_2))i(u(\gamma_1\gamma_2)) \\ &= \sigma(\rho(\gamma_1))i(u(\gamma_1))\sigma(\rho(\gamma_2))i(u(\gamma_2)) \\ &= \widetilde{\rho}(\gamma_1)\widetilde{\rho}(\gamma_2), \end{aligned}$$

whence $\widetilde{\rho}$ is indeed a homomorphism. This shows in particular that ρ lifts whenever $\rho^*e = 0$. We leave the converse implication to the reader. □

Corollary 2.8 (Killing H^2 via central extensions)**.** *Let G be a group and $e \in H^2(G; R)$. Then there exists a central extension ξ as in* (2.1) *such that $p^*e = 0 \in H^2(\widetilde{G}; R)$.*

Proof. Choose ξ so that $e = e(\xi)$. Then

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r] & \widetilde{G} \arrow[r, "p"] & G \arrow[r] & 1 \\ & & & \widetilde{G} \arrow[ul, "1_{\widetilde{G}}"] \arrow[u, "p"'] & \end{tikzcd},$$

commutes, whence $p^*e = 0$. □

2.6. **The case of surface groups.** We now specialize the above results in the case where $G = \Gamma_g$ is a surface group. From the topological model we know that $H^2(\Gamma_g; \mathbb{Z}) \cong \mathbb{Z}$. By Proposition 2.6 there exists thus for every integer n a central $\mathbb{Z}$-extension $\widetilde{\Gamma_g^{(n)}}$ such that these exhaust all equivalence classes of central $\mathbb{Z}$-extensions. The extension $\widetilde{\Gamma_g^{(0)}}$ is just the trivial extension $\Gamma \times \mathbb{Z}$. In general we have presentations

$$\widetilde{\Gamma_g^{(n)}} = \langle a_1, b_1, \ldots, a_g, b_g, c \,|\, \prod_{j=1}^{g} [a_j, b_j] = z^n,\ [a_j, c] = [b_j, c] = e \rangle,$$

where the kernel of the map $\widetilde{\Gamma_g^{(n)}} \to \Gamma_g$ is generated by c. Since $H_b^2(G, \mathbb{R})$ is cyclic, a special role is played by the central extension

$$p_{\Gamma_g} : \widetilde{\Gamma_g} \to \Gamma_g,$$

where $\widetilde{\Gamma_g} := \widetilde{\Gamma_g^{(1)}}$. This central extension is universal in the following sense:

Proposition 2.9 (Universal central extensions of surface groups)**.** *Let $p_G : \widetilde{G} \to G$ be a central $\mathbb{Z}$-extension and $\rho : \Gamma_g \to G$ an arbitrary homomorphism. Then there exists a lift $\widetilde{\rho} : \widetilde{\Gamma_g} \to \widetilde{G}$ making the diagram*

$$\begin{tikzcd} \widetilde{\Gamma_g} \arrow[r, dotted, "\widetilde{\rho}"] \arrow[d, "p_{\Gamma_g}"'] & \widetilde{G} \arrow[d, "p_G"] \\ \Gamma_g \arrow[r, "\rho"] & G \end{tikzcd}$$

commute.

Proof. Let $\alpha \in H^2(G;\mathbb{Z})$ be the class associated with the extension p_G and $\beta := \rho^*\alpha$. Then there exists $n \in \mathbb{Z}$ such that β is represented by the extension $p_n : \widetilde{\Gamma_g^{(n)}} \to \Gamma_g$. Now there is a homorphism $\widetilde{\Gamma} \to \widetilde{\Gamma_g^{(n)}}$ given by $c \mapsto c^n$, hence we get

$$\begin{tikzcd}
\widetilde{\Gamma_g} \arrow[d] \arrow[dr, dotted, "\widetilde{\rho}"] \arrow[dd, bend right=60] & \\
\widetilde{\Gamma_g^{(n)}} \arrow[r] \arrow[d, "p_{\Gamma_g}"'] & \widetilde{G} \arrow[d, "p_n"] \\
\Gamma_g \arrow[r, "\rho"] & G.
\end{tikzcd}$$

□

2.7. **Actions on the circle.** We provide a first application of the methods developed so far. Let Γ be a group acting on the circle by orientation-preserving homomorphisms. When does this action lift to an action on the real line?

To answer this question, consider the group $G := \mathrm{Homeo}^+(S^1)$ of orientation-preserving homeomorphisms of the circle and its central extension

$$\xi_{S^1} = (0 \to \mathbb{Z} \to \widetilde{G} \to G \to 1),$$

where

$$\widetilde{G} := \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) = \{f \in \mathrm{Homeo}^+(\mathbb{R}) \mid f(x+1) = f(x) + 1\}.$$

The class $e(S^1) := e(\xi_{S^1})$ is called the *Euler class of the circle*. Now an orientation-preserving action of Γ on the circle is the same as a homomorphism $\rho : \Gamma \to G$. We refer to $\rho^*(e(S^1)) \in H^2(\Gamma)$ as the Euler class of this circle action. Then we obtain the following special case of Proposition 2.7:

Corollary 2.10. *An action of* Γ *on the circle lifts to an action on the real line if and only if the Euler class of the circle action vanishes.*

3. Bounded cohomology I: Elementary theory

3.1. **Definition and models.** Bounded group cohomology was introduced independently by Traubel (working on groups, unpublished) and Johnson ([27], working on Banach algebras) and popularized by Gromov in his famous 1982 paper [23]. It is an important tool in modern group theory ever since. As for classical cohomology, we can give a functorial, a topological and a combinatorial definition. The combinatorial definition is the easiest to state, so we start with this one:

Recall that the homogeneous standard resolution, say with integer coefficients, is formed by the modules $C^n(G;\mathbb{Z}) := \mathrm{Map}(G^{n+1}, \mathbb{Z})$ and homogeneous differentials d. If we define $C^n_b(G;\mathbb{Z}) := l^\infty(G^{n+1}, \mathbb{Z})$, where $l^\infty(-;\mathbb{Z})$ denotes bounded $\mathbb{Z}$-valued functions, then $(C^n_b(G;\mathbb{Z}), d)$ is a subcomplex of $(C^n(G;\mathbb{Z}), d)$ and we define:

Definition 3.1. The cohomology $H^\bullet_b(G;\mathbb{Z}) := H^\bullet(C^n_b(G;\mathbb{Z}), d)$ is called the *bounded integral group cohomology* of G.

As in the classical case, we also have an inhomogeneous model

$$H_b^\bullet(G;\mathbb{Z}) \cong H^\bullet(l^\infty(G^0;\mathbb{Z}) \to l^\infty(G^1;\mathbb{Z}) \to l^\infty(G^2;\mathbb{Z}) \to l^\infty(G^3;\mathbb{Z}) \to \dots).$$

Bounded group cohomology with *real coefficients* is defined accordingly, replacing $\mathbb{Z}$ by $\mathbb{R}$ everywhere. Integral and real bounded group cohomology are related by the *Gersten exact sequence*

$$\dots \to H_b^n(G;\mathbb{Z}) \to H_b^n(G;\mathbb{R}) \to H^n(G;\mathbb{R}/\mathbb{Z}) \to H_b^{n+1}(G;\mathbb{Z}) \to \dots$$

We will mostly be working with real coefficients in our applications, but occasionally passing to integral coefficients is necessary.

Let us now turn to other models of bounded cohomology: One can define bounded singular cohomology of a topological space X by considering only those integral (or real-valued) cochains, which considered as functions on singular simplices are bounded. If we denote the group of this cochains by $C_b^n(X;\mathbb{Z})$ (or $C_b^n(X;\mathbb{R})$) then

$$H_b^\bullet(X;\mathbb{Z}) := H^\bullet(C_b^\bullet(X;\mathbb{Z}), d), \quad H_b^\bullet(X;\mathbb{R}) := H^\bullet(C_b^\bullet(X;\mathbb{Z}), d).$$

Now as in the classical case one shows that $H_b^\bullet(G;\mathbb{Z}) \cong H_b^\bullet(BG;\mathbb{Z})$ and $H_b^\bullet(G;\mathbb{R}) \cong H_b^\bullet(BG;\mathbb{R})$. However, actually much more is true: Bounded cohomology does not see higher homotopy groups, whence:

Theorem 3.2 (Gromov-Brooks-Ivanov, cf. [23, 26])**.** *Let X be a space which has the homotopy type of a countable CW-complex. Then*

$$H_b^\bullet(X;\mathbb{Z}) \cong H_b^\bullet(\pi_1(X);\mathbb{Z}), \quad H_b^\bullet(X;\mathbb{R}) \cong H_b^\bullet(\pi_1(X);\mathbb{R}).$$

Finally, there is also a categorical interpretation of $H_b^\bullet(G;\mathbb{R})$, which was obtained by Bühler in his thesis [3]. Namely, for $H_b^\bullet(G;-)$ is the derived functor of $(-)^G$ in the (semi-abelian) category of Banach-G-modules. We will not go into this abstract definition any further; it basically means that, like usual cohomology, bounded cohomology can be computed from many different resolutions. One such resolution, the so-called boundary resolution of Burger and Monod, will be discussed below. To the best of our knowledge, there is no categorical approach to integral bounded group cohomology.

In the remainder of this section we present some standard material concerning bounded cohomology. Unless otherwise mentioned, this material can be found in [29, 3, 12].

3.2. **The comparison map I: Non-surjectivity for amenable groups.** The inclusion of subcomplexes $C_b^n(G;\mathbb{R}) \hookrightarrow C^n(G;\mathbb{R})$ induces, on the level of cohomology, a *comparison map*

$$c^\bullet = c_G^\bullet : H_b^\bullet(G;\mathbb{R}) \to H_b^\bullet(G;\mathbb{R}),$$

which *in general is neither injective nor surjective.*[1] Let us first explain, why it fails to be surjective in general. Digressing a little bit, let us first point out that the usual group cohomology $H^\bullet(F;\mathbb{R})$ vanishes for any finite group F. Indeed, by averaging

[1]As pointed out to me by M. Bucher, this half-sentence seems to be contained in every single paper on bounded cohomology.

over the finite group we obtain equivariant operators $A : C^n(G;\mathbb{R}) \to C^{n-1}(G;\mathbb{R})$, which satisfy $dA = \mathrm{Id}$. These then provide a contracting homotopy for the complex

$$\begin{tikzcd} 0 \arrow[r] & C^0(G;\mathbb{R})^G \arrow[r, "d"] & C^1(G;\mathbb{R})^G \arrow[r, "d"] \arrow[l, bend left, "A"] & C^2(G;\mathbb{R})^G \arrow[r, "d"] \arrow[l, bend left, "A"] & C^3(G;\mathbb{R})^G \arrow[r, "d"] \arrow[l, bend left, "A"] & \dots \arrow[l, bend left, "A"] \end{tikzcd},$$

showing that $H^n(F;\mathbb{R}) = \{0\}$ for $n > 0$. This argument works only for finite groups, since these are the only ones admitting an equivariant averaging operator. However, in bounded cohomology we can generalize the above argument quite a bit. Recall that a group G is called *amenable* if $l^\infty(G)$ admits a G-invariant *mean*, i.e. a linear fuctional $\mathfrak{m} \in l^\infty(G)'$ such that $\mathfrak{m}(1_G) = 1$, $\|\mathfrak{m}\| = 1$ and $f \geq 0 \Rightarrow \mathfrak{m}(f) \geq 0$. Using such a mean it is easy to construct averaging operators $A : C_b^n(G;\mathbb{R}) \to C_b^{n-1}(G;\mathbb{R})$ which provide a contracting homotopy

$$\begin{tikzcd} 0 \arrow[r] & C_b^0(G;\mathbb{R})^G \arrow[r, "d"] & C_b^1(G;\mathbb{R})^G \arrow[r, "d"] \arrow[l, bend left, "A"] & C_b^2(G;\mathbb{R})^G \arrow[r, "d"] \arrow[l, bend left, "A"] & C_b^3(G;\mathbb{R})^G \arrow[r, "d"] \arrow[l, bend left, "A"] & \dots \arrow[l, bend left, "A"] \end{tikzcd}.$$

We thus deduce:

Proposition 3.3. *If G is amenable, then $H_b^n(G;\mathbb{R}) = \{0\}$ for all $n > 0$.*

We will not have the time to discuss amenable groups in detail; let us just remark that

- $\mathbb{Z}$ and more generally, abelian groups, are amenable by the Markov-Kakutani fixed point theorem;
- finite groups are (obviously) amenable;
- extensions and directed unions of amenable groups are amenable;
- combining these three examples, directed union of finite-by-solvable groups are amenable; these are called *elementary amenable*;
- finitely generated groups of subexponential growth are amenable.

It was a long standing open question, whether every amenable group is elementary amenable. The question was answered in the negative by constructing non-elementary amenable groups of subexponential (in fact, intermediate) growth. The first such examples were constructed by Grigorchuk in 1980.

After this short digression we now return to our comparison map. From the amenability of $\mathbb{Z}^n$ we deduce that the comparison map is not surjective in any degree > 0:

Corollary 3.4. *For $n > 0$ the comparison map*

$$c_{\mathbb{Z}}^n : \{0\} \cong H_b^n(\mathbb{Z}^n;\mathbb{R}) \to H^n(\mathbb{Z}^n;\mathbb{R}) \cong \mathbb{R}$$

is not surjective.

3.3. **The comparison map II: Non-injectivity and quasimorphisms.** More surprising than the non-surjectivity of the comparison map is probably the non-injectivity. To understand what is going on here, we have to unravel definitions. We are going to work in the inhomogeneous bar resolution, and we will focus on low degrees. In degrees 0 and 1 there is nothing to gain: As in Exercise 2.4 one shows that for any group G one has $H_b^0(G;\mathbb{R}) \cong \mathbb{R}$ and $H_b^1(G;\mathbb{R}) = \mathrm{Hom}_b(G;\mathbb{R})$, the space of bounded homomorphisms into $\mathbb{R}$. However, $\mathbb{R}$ does not have bounded subgroups, whence:

Lemma 3.5. *For any group G we have $H_b^1(G;\mathbb{R}) = \{0\}$.*

Thus in order to find comparison maps with non-trivial kernel we have to go at least to degree 2. Here the relevant part of the inhomogeneous bar resolution is given by

$$\begin{array}{ccccc} \mathrm{Map}(G;\mathbb{R}) & \xrightarrow{\partial^1} & \mathrm{Map}(G^2;\mathbb{R}) & \xrightarrow{\partial^2} & \mathrm{Map}(G^3;\mathbb{R}) \\ \uparrow & & \uparrow & & \uparrow \\ l^\infty(G;\mathbb{R}) & \xrightarrow{\partial^1} & l^\infty(G^2;\mathbb{R}) & \xrightarrow{\partial^2} & l^\infty(G^3;\mathbb{R}) \end{array}$$

where $\partial^1 f(g,h) = f(h) - f(gh) + f(h)$. Now assume that

$$\alpha = [c] \in EH_b^2(G;\mathbb{R}) := \ker(c_G^2).$$

Since $c_G^2(\alpha) = 0$ there exists $f \in \mathrm{Map}(G;\mathbb{R})$ such that $\partial^1 f = c$. Note that

$$\sup_{g,h\in G} |f(gh) - f(g) - f(h)| = \sup_{g,h\in G} |c(g,h)| < \infty.$$

Thus f is almost a homomorphism. We introduce a name for such functions:

Definition 3.6. A function $f : G \to \mathbb{R}$ is called a *quasimorphism* of *defect* $D(f)$ if

$$D(f) := \sup_{g,h\in G} |f(gh) - f(g) - f(h)| < \infty.$$

Two quasimorphisms f_1, f_2 are called *equivalent* if $\|f_1 - f_2\|_\infty < \infty$. We denote by $\widetilde{\mathcal{Q}}(G)$ the space of quasimorphisms on G and by $\mathcal{Q}(G)$ the space of equivalence classes of quasimorphisms.

By the previous considerations we have a surjective linear map

$$q : \widetilde{\mathcal{Q}}(G) \to EH_b^2(G;\mathbb{R}), \quad f \mapsto [df].$$

Exercise 3.7. Show that the kernel of q is given by $l^\infty(G) \oplus \mathrm{Hom}(G;\mathbb{R})$.

As an immediate consequence of the exercise we have:

Proposition 3.8. *There map q induces an isomorphism*

$$\frac{\mathcal{Q}(G)}{\mathrm{Hom}(G;\mathbb{R})} = \frac{\widetilde{\mathcal{Q}}(G)}{l^\infty(G) \oplus \mathrm{Hom}(G;\mathbb{R})} \cong EH_b^2(G;\mathbb{R}).$$

Since it is slightly inconvenient to work with equivalence classes of quasimorphisms all the time, it is useful to find canonical representatives.

Proposition 3.9. *Let f be a quasimorphism.*

(i) *For every $g \in G$ the limit*

$$\widetilde{f}(g) := \lim_{n\to\infty} \frac{f(g^n)}{n}$$

exists.

(ii) *$\widetilde{f}$ is a quasimorphism, which is equivalent to f.*

(iii) *$\widetilde{f}$ is homogeneous, i.e. $f(g^n) = n \cdot f(g)$ for all $n \geq 0$.*

(iv) $\widetilde{f}$ *is the unique homogeneous quasimorphism equivalent to* f.

Exercise 3.10. Prove Proposition 3.9.

In view of the proposition, we can identify $\mathcal{Q}(G)$ with the space of *homogeneous* quasimorphisms on G. Thus:

Corollary 3.11. *Let* G *be a group. Then* c^2_G *is injective if and only if every homogeneous quasimorphism on* G *is a homomorphism.*

Combining this with Proposition 3.3 we deduce:

Corollary 3.12. *If* G *is an amenable group, then every homogeneous quasimorphism on* G *is a homomorphism.*

The following special case is noteworthy:

Corollary 3.13. *Let* $f : G \to \mathbb{R}$ *be a homogeneous quasimorphism. Let* $a, b \in G$ *and assume that* a *and* b *commute. Then* $f(ab) = f(a) + f(b)$.

Proof. Let $H := \langle a, b\rangle$. Then H is abelian, hence amenable, and thus $f|_H$ is a homomorphism. □

The following lemma collects further properties of homogeneous quasimorphisms:

Lemma 3.14. *Let* $f : G \to \mathbb{R}$ *be a homogeneous quasimorphism. Then the following hold:*

(i) $f(g^{-1}) = -f(g)$.
(ii) $f(g^n) = n \cdot f(g)$ *for all* $n \in \mathbb{Z}$.
(iii) f *is conjugation-invariant.*
(iv) *If* $N \lhd G$ *is normal,* $p : G \to G/N$ *is the canonical projection and* $f|_N = 0$, *then there exists a homogeneous quasimorphism* $F : N \to \mathbb{R}$ *with* $f = F \circ p$.
(v) f *is a homomorphism if and only if* $f|_{[G,G]} = 0$.

Proof. (i) $0 = f(e) = f(g^n g^{-n}) = f(g^n) + f((g^{-1})^n) + o(1) = n \cdot f(g) + n \cdot f(g^{-1}) + o(1)$. Now divide by n and let $n \to \infty$. (ii) immediate from (i). (iii) Compute

$$n \cdot f(hgh^{-1}) = f(hg^n h^{-1}) = f(h) + f(g^n) + f(h^{-1}) + o(1) = n \cdot f(g) + o(1),$$

divide by n and let $n \to \infty$. (iv) For every coset gN pick a representative g and set $F_0(gN) := f(g)$. It is easy to check that F_0 is a quasimorphism. Let F be the homogenization of F_0; then one checks that $p^*F - f$ is a bounded homogeneous function, hence equal to 0. (v) If f is a homomorphism, then clearly $f|_{[G,G]} = 0$. Conversely, assume that $f|_{[G,G]} = 0$ and denote by $G_{\mathrm{ab}} := G/[G, G]$ the abelianization. By (iv), we then find a homogeneous quasimorphism $F : G_{\mathrm{ab}} \to \mathbb{R}$ with $f = F \circ p$, where $p : G \to G_{\mathrm{ab}}$ is the canonical projection. Now by Corollary 3.12, F is a homomorphism, hence f is a homomorphism as well. □

We now return to the problem of (non-) injectivity of the comparison map.

Example 3.15 (Brooks)**.** Let $G = F_2$ be the free group with basis $\{a, b\}$. Identify elements of G with reduced words over $\{a, b\}$. Given such a reduced word w, denote

by $\#_{ab}(w)$ the number of occurrences of the sequence ab in w, and define $\#_{(ab)^{-1}}(w)$ similarly. Then
$$\varphi_{ab} : F_2 \to \mathbb{Z}, \quad w \mapsto \#_{ab}(w) - \#_{(ab)^{-1}}(w)$$
is a quasimorphism of defect 1.

Corollary 3.16. *$c^2_{F_2}$ is not injective.*

Proof. The homogenization $f := \widetilde{\varphi}_{ab}$ vanishes on a and b and satisfies $f(ab) = 1$. If $c^2_{F_2}$ was injective, then f was a homomorphism by Corollary 3.11. However, there is no homomorphism f of F_2 with $f(a) = f(b) = 0$ and $f(ab) = 1$. □

With (quite) a bit more work one can push this idea to show that, in fact [22]
$$\dim H^2_b(F_2; \mathbb{R}) = \dim EH^2_b(F_2; \mathbb{R}) = \infty.$$
The obvious idea is to construct counting quasimorphisms φ_w for arbitrary $w \in F_2$. What is not obvious, is that sufficiently many of these quasimorphisms are linearly independent. This was established by Grigorchuk in 1994, after several wrong proofs had appeared.

3.4. **Aside: A geometric disaster.** The most powerful tool in cohomology of topological spaces is arguably the excision theorem. It says that if $X = U \cup V$ is a space build from subspaces U and V, then $H^\bullet(X; \mathbb{R})$ can be computed from the cohomologies of U, V and $U \cap V$. For example, if X is the figure eight, then we can decompose it into two circles U and V, intersecting in a point. Thus the cohomology of the figure eight can be computed from the cohomology of the circle and the cohomology of a point. This is true for all generalized cohomology theories in the sense of Eilenberg-Steenrod, even for more exotic ones like K-theory. Now, for bounded cohomology this excision theorem fails in the most horrible way. For the circle S^1 we compute
$$H^n_b(S^1; \mathbb{R}) = H^n_b(\mathbb{Z}; \mathbb{R}) = \{0\} \quad (n \geq 1).$$
However, for the figure eight we have
$$H^n_b(8; \mathbb{R}) = H^n_b(F_2; \mathbb{R}),$$
which is not only non-zero, but in fact infinite-dimensional in degree 2. This is the reason why bounded cohomology of topological spaces has evaded almost any computational attempts beyond degree 2 .

3.5. **An interpretation of H^2_b.** We have seen that $H^2(G; \mathbb{R})$ classifies central $\mathbb{R}$-extensions of G. What does $H^2_b(G; \mathbb{R})$ classify? Since H^2 classifies central extensions and EH^2_b classifies quasimorphisms, it is reasonable to expect that H^2_b classifies a combination of the two. This is indeed the case, although it has not been spelled out explicitly in the literature. We suggest here the following new terminology;

Definition 3.17. A *quasi-corner* over G is a diagram
$$\begin{tikzcd} \widetilde{G} \arrow[r, "f"] \arrow[d, "p"] & \mathbb{R} \\ G & \end{tikzcd}$$

such that p is onto, $\ker(p)$ is central in $\widetilde{G}$ and f is a homogeneous quasimorphism.

We write $C(f,p)$ to denote the above quasi-corner. Note that every quasimorphism f on G defines a quasi-corner $C(f, \mathrm{id}_G)$. This allows us to consider quasimorphism as special quasi-corners. We now introduce a notion of equivalence of quasi-corners, which generalizes the notion of being cohomologous for quasimorphisms. For this let $C(f_1,p_1)$ and $C(f_2,p_2)$ be quasi-corners as above. Let $\widetilde{G}$ be the pushout

$$\begin{tikzcd} & \widetilde{G} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, "p"] & \\ \widetilde{G_1} \arrow[dr, "p_1"'] & & \widetilde{G_2} \arrow[dl, "p_2"] \\ & G & \end{tikzcd}$$

where $p := p_1 \circ \pi_1 = p_2 \circ \pi_2$. Then p is surjective, and since the kernels of p_1 and p_2 are abelian, its kernel is amenable. In other words, $p : \widetilde{G} \to G$ is an (in general non-central) amenable extension.

Definition 3.18. The quasi-corners $C(f_1,p_1)$ and $C(f_2,p_2)$ are *equivalent* if

$$(\pi_1^* f_1 - \pi_2^* f_2)|_{[\widetilde{G},\widetilde{G}]} \equiv 0.$$

We denote by $\mathcal{QC}(G)$ the set of equivalence classes of quasi-corners over G.

Note that in view of Proposition 3.14 the above condition is equivalent to $\pi_1^* f_1 - \pi_2^* f_2$ being a homomorphism. We now relate equivalence classes of quasi-corners over G to classes in $H_b^2(G;\mathbb{R})$; for this the key observation as as follows:

Proposition 3.19 (Invariance under amenable extensions)**.** *For any amenable extension $p : \widetilde{G} \to G$ the map $p^* : H_b^2(G;\mathbb{R}) \to H_b^2(\widetilde{G};\mathbb{R})$ is an isomorphism. In particular, this is the case for any central extension.*

Proof. Let us denote by C the kernel of p so that we have a short exact sequence

$$\{e\} \to C \xrightarrow{i} \widetilde{G} \xrightarrow{p} G \to \{e\},$$

which we can exploit cohomologically. Indeed, a standard tool in general homological algebra is the five-term exact sequence of a derived functor. In the case of bounded cohomology this five-term exact sequence for the central extension ξ reads[2]

$$0 \to H_b^2(G;\mathbb{R}) \xrightarrow{p^*} H_b^2(\widetilde{G};\mathbb{R}) \to H_b^2(C;\mathbb{R}) \to H_b^3(G;\mathbb{R}) \to H_b^3(\widetilde{G};\mathbb{R}).$$

Now C is assumed amenable, hence the proposition follows from Proposition 3.3. □

[2]If you know classical group cohomology, you might know a similar sequence starting from H^1; in bounded cohomology the degree is shifted by one, since H_b^1 is always trivial. The proof (essentially a spectral sequence argument) is the same.

Given a quasi-corner $C = C(f, p)$ we define

$$\alpha_b(C) := (p^*)^{-1}(df) \in H_b^2(G; \mathbb{R}).$$

We then say that C *realizes* the class $\alpha_b(C)$. We observe:

Proposition 3.20. *If two quasi-corners are equivalent, then they realize the same bounded cohomology class.*

Proof. This is immediate from Propositions 3.19, 3.8 and 3.9. □

We thus obtain a well-defined map

$$\alpha : \mathcal{QC}(G) \to H_b^2(G; \mathbb{R}), \quad [C] \mapsto \alpha_b(C).$$

Proposition 3.21. *The map $\alpha : \mathcal{QC}(G) \to H_b^2(G; \mathbb{R})$ is a bijection.*

Proof. In view of Proposition 3.14.(v), injectivity follows again from Propositions 3.19, 3.8 and 3.9. Indeed, two quasi-corners $C(f_1, p_1)$ and $C(f_2, p_2)$ represent the same class in bounded cohomology if and only if $[d\pi_1^* f_1] = [d\pi_2^* f_2]$, which by Proposition 3.14.(v) precisely means that $\pi_1^* f_1 - \pi_2^* f_2$ is a homomorphism. By definition this means that the two quasi-corners are equivalent. It remains to show that every bounded cohomology class $\alpha_b \in H_b^2(G; \mathbb{R})$ can be realized by a quasi-corner. For this let $\alpha := c_G^2(\alpha_b) \in H^2(G; \mathbb{R})$. By Corollary 2.8 there exists a central extension ξ of G, along which the pullback of α vanishes. (In general, ξ will be an $\mathbb{R}$-extension, but in many favorable situations e.g. if α is rational we can actually pick a $\mathbb{Z}$-extension.) We now fix one such central extension

$$\xi = (0 \to C \to \widetilde{G} \xrightarrow{p} G \to 1),$$

where C is some subgroup of $\mathbb{R}$ and $p^*\alpha = 0$. By naturality of the comparison map we now have

$$c_{\widetilde{G}}^2(p^*(\alpha_b)) = p^* c_{\widetilde{G}}^2(\alpha_b) = p^*\alpha = 0,$$

whence

$$p^*\alpha_b \in EH_{cb}^2(G; \mathbb{R}).$$

By Proposition 3.8 and Proposition 3.9 this implies that there exists a homogeneous quasimorphism $f : \widetilde{G} \to \mathbb{R}$ such that

$$df = p^*\alpha_b,$$

whence $\alpha_b = \alpha_b(C(f, p))$. □

The reason why the quasi-corner model for $H_b^2(G; \mathbb{R})$ is useful is the following naturality property, which we will use many times:

Exercise 3.22. Show that the obvious pullback of quasi-corners (descends to equivalence classes and) induces pullback on the level of bounded cohomology.

3.6. **The case of surface groups.** In the case of surface groups the situation is considerably simplified by the existence of a universal central extension $p_{\Gamma_g} : \widetilde{\Gamma_g} \to \Gamma_g$ as defined in Section 2.6. Indeed, assume that we are given two quasi-corners

$$\begin{tikzcd} \widetilde{G} \arrow[r, "f_G"] \arrow[d, "p_G"] & \mathbb{R} \\ G & \end{tikzcd} \qquad\qquad \begin{tikzcd} \widetilde{H} \arrow[r, "f_H"] \arrow[d, "p_H"] & \mathbb{R} , \\ H & \end{tikzcd}$$

with $\ker(p_G) \cong \ker(p_H) \cong \mathbb{Z}$, representing classes α_b^G and α_b^H respectively. Assume moreover that we are given homomorphisms $\rho_G : \Gamma \to G$ and $\rho_H : \Gamma \to H$. By Proposition 2.9 we then find lifts making the diagram

$$\begin{tikzcd} \mathbb{R} & \widetilde{G} \arrow[l, "f_G"'] \arrow[d, "p_G"'] & \widetilde{\Gamma_g} \arrow[l, "\widetilde{\rho_G}"'] \arrow[r, "\widetilde{\rho_H}"] \arrow[d, "p_{\Gamma_g}"'] & \widetilde{H} \arrow[r, "f_H"] \arrow[d, "p_H"] & \mathbb{R} \\ & G & \Gamma_g \arrow[l, "\rho_G"] \arrow[r, "\rho_H"'] & H & \end{tikzcd} \tag{3.1}$$

commute. Then we have the following special case of Proposition 3.21:

Corollary 3.23. *In the situation of* (3.1) *the following are equivalent:*

(i) $\rho_G^*\alpha_b^G = \rho_H^*\alpha_b^H \in H_b^2(\Gamma;\mathbb{R})$.
(ii) *The function* $f_G \circ \widetilde{\rho_G} - f_H \circ \widetilde{\rho_H} : \widetilde{\Gamma_g} \to \mathbb{R}$ *is a homomorphism.*
(iii) $(f_G \circ \widetilde{\rho_G} - f_H \circ \widetilde{\rho_H})|_{[\widetilde{\Gamma_g},\widetilde{\Gamma_g}]} \equiv 0$.

4. Bounded Euler class and applications to surface group representations

Having developed the most elementary parts of the theory of bounded cohomology we now turn to a first application. We introduce the bounded Euler class and prove a special case of a famous theorem of Ghys [19, 20].

4.1. **The bounded Euler class and the translation number.** At this point we have assembled enough information about bounded cohomology to give a first application. For this we return to the situation of Section 2.7 and consider the central extension $p : \widetilde{G} \to G$ given by $G := \mathrm{Homeo}^+(S^1)$, $\widetilde{G} := \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$. Recall that this central extension corresponds to a class $e(S^1) \in H^2(G;\mathbb{Z})$, which is represented by the cocycle

$$c_\sigma(g,h) = (\sigma(g)\sigma(gh)^{-1}\sigma(g))(0).$$

A specific section $\sigma : G \to \widetilde{G}$ can be given as follows: Let $\sigma(f) \in \widetilde{G}$ be the unique lift of f with $\sigma(f)(0) \in [0,1)$. Then, obviously, c_σ is bounded, hence defines a class

$$e_b(S^1) := [c_\sigma(g,h)] \in H_b^2(G;\mathbb{Z}).$$

Note that by definition, $e_b(S^1)$ is mapped to $e(S^1)$ under the comparison map, hence it is called a *bounded Euler class*. We also denote by $e_b^{\mathbb{R}}(S^1)$ the corresponding class in $H_b^2(G;\mathbb{R})$. We will usually shorten the notation to $e, e_b, e^{\mathbb{R}}, e_b^{\mathbb{R}}$, dropping the S^1 from the notation.

Now the *real bounded Euler class* $e_b^{\mathbb{R}}(S^1)$ can be represented by a quasi-corner of the form

$$\begin{tikzcd} \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) \arrow[r, "T"] \arrow[d, "p"] & \mathbb{R} \\ \mathrm{Homeo}^+(S^1), & \end{tikzcd}$$

and since $\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$ is perfect, the homogeneous quasimorphism $T : \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) \to \mathbb{R}$ is actually uniquely determined.

Proposition 4.1. (i) *For every $x \in \mathbb{R}$ the map $T_x : \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) \to \mathbb{R}$ given by*

$$T_x(f) = f(x) - x$$

is a quasimorphism.

(ii) *The quasimorphisms T_x are at mutually bounded distance, hence have a common homogenization $T : \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) \to \mathbb{R}$ given by*

$$Tf = \lim_{n\to\infty} \frac{f^n(x) - x}{n},$$

which is independent of $x \in \mathbb{R}$.

(iii) *If $p : \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) \to \mathrm{Homeo}^+(S^1)$ is the canonical projection, then $e_b^{\mathbb{R}}(S^1)$ is represented by the quasi-corner $C(-T, p)$.*

The quasimorphism T was discovered by Poincaré in 1889. It is called the *translation number quasimorphism* and plays a major role in the theory of dynamical systems.

Proof of Proposition 4.1. (i), (ii) The key observation is that for $f, g \in \widetilde{G}$ and $x \in \mathbb{R}$ we have

$$|T_x(f) - T_y(f)| = |(f(x) - x) - (f(y) - y)| \quad \leq \quad 1.$$

Indeed, (ii) is immediate from this observation, and (i) also follows since

$$T_x(f \circ g) - T_x(f) - T_x(g) = T_{g(x)}(f) - T_x(f).$$

It remains to show (iii). We will actually show the following more precise statement: If $\sigma : G \to \widetilde{G}$ is the unique section with $\sigma(f)(0) \in [0, 1)$ and e_σ is the corresponding cocycle representative of the Euler class of the extension p, then $p^*e_\sigma = dA$, where $A(f) = \lfloor f(0) \rfloor$. Since $|A(f) - T_0(f)| < 1$, this implies the statement. The key step in the proof is to show that for $f_1, f_2 \in \widetilde{G}$ we have

$$\lfloor (f_1f_2)(0) \rfloor = -\lfloor (f_1f_2)^{-1}(0) \rfloor. \tag{4.1}$$

Indeed, every $f \in \widetilde{G}$ satisfies

$$\lfloor f^{-1}\lfloor f(0) \rfloor \rfloor \leq \lfloor f^{-1}(f(0)) \rfloor = 0, \quad \lfloor f^{-1}\lfloor f(0) \rfloor \rfloor > \lfloor f^{-1}(f(0) - 1) \rfloor = -1,$$

leading to $\lfloor f^{-1}\lfloor f(0) \rfloor \rfloor = 0$. This now yields

$$0 = \lfloor f^{-1}(\lfloor f(0) \rfloor) \rfloor = \lfloor f^{-1}(0 + \lfloor f(0) \rfloor) \rfloor = \lfloor f^{-1}(0) + \lfloor f(0) \rfloor \rfloor = \lfloor f^{-1}(0) \rfloor + \lfloor f(0) \rfloor,$$

which for $f := f_1 f_2$ yields (4.1). Thus if we denote $\overline{f} := f - A(f)$, then

$$\begin{aligned} (p^*e_\sigma)(f_1, f_2) &= i^{-1}(\sigma(p(f_1)p(f_2))^{-1}\sigma(p(f_1))\sigma(p(f_2))) \\ &= (\overline{(f_1f_2)^{-1}}\bar{f}_1\bar{f}_2)(0) \end{aligned}$$

$$
\begin{aligned}
&= \quad (\overline{(f_1f_2)^{-1}}\bar{f}_1(f_2 - \lfloor f_2(0)\rfloor)(0)\\
&= \quad (\overline{(f_1f_2)^{-1}}(f_1f_2 - \lfloor f_2(0)\rfloor - \lfloor f_1(0)\rfloor)(0)\\
&= \quad 0 - \lfloor f_2(0)\rfloor - \lfloor f_1(0)\rfloor - \lfloor (f_1f_2)^{-1}(0)\rfloor\\
&\overset{(4.1)}{=} \quad \lfloor (f_1f_2)(0)\rfloor - \lfloor f_2(0)\rfloor - \lfloor f_1(0)\rfloor)\\
&= \quad -dA(f_1, f_2).
\end{aligned}
$$

□

4.2. **Vanishing of bounded Euler classes.** Let G be a group and $\rho : G \to \mathrm{Homeo}^+(S^1)$ be a circle action. We have seen that $\rho^*e = 0$ if and only if the action lifts to the real line. What is the meaning of $\rho^*e_b = 0$? This condition of course implies that the action lifts, however, since the comparison map of G may have a huge kernel (as we have seen in the case of free groups), there is much more information to be gained from e_b.

Proposition 4.2. *Let $\rho : G \to \mathrm{Homeo}^+(S^1)$ be a circle action with $\rho^*e_b = 0$. Then $\rho(G)$ fixes a point on S^1.*

This is a baby version of a famous theorem of Ghys [19, 20], which says that ρ^*e_b is a complete quasi-conjugacy invariant of circle actions. The short and elementary proof given below is due to M. Bucher and the author. The general theorem is based on a similar idea, but much more technical.

Proof. Let $u : G \to \mathbb{Z}$ be a bounded function with $\rho^*e_\sigma = du$, where σ is the special section defined above. By Proposition 2.7 we have a homomorphism

$$\widetilde{\rho} : G \to \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}), \quad \widetilde{\rho}(g) = \sigma(\rho(g)) \cdot i(u(g)).$$

In particular,

$$\widetilde{\rho}(g)(0) = \sigma(\rho(g))(0) + u(g).$$

Now, by choice of our section, $\sigma(\rho(g))(0) \in [0, 1)$, hence $\widetilde{\rho}(g)(0)$ is bounded. Set

$$F^+(\widetilde{\rho}) := \sup_{g\in G} \widetilde{\rho}(g)(0);$$

then $F^+(\widetilde{\rho})$ is a fixed point for $\widetilde{\rho}(G)$, hence its image in S^1 is a fixed point for $\rho(G)$. □

We will also need a version of the proposition for the *real* bounded Euler class:

Corollary 4.3. *Let $\rho : G \to \mathrm{Homeo}^+(S^1)$ be a circle action with $\rho^*e_b = 0$. Then $\rho([G, G])$ fixes a point on S^1.*

Proof. Consider the following segment of the Gersten sequence for G:

$$H^1(G, \mathbb{R}/\mathbb{Z}) = \mathrm{Hom}(G, \mathbb{R}/\mathbb{Z}) \xrightarrow{\delta} H^2_b(G, \mathbb{Z}) \longrightarrow H^2_b(G, \mathbb{R}) .$$

Then our assumption $\rho^*(e_b^{\mathbb{R}}) = 0$ implies that there exists a homomorphism $\chi : G \to \mathbb{R}/\mathbb{Z}$ such that $\rho^*(e_b) = \delta(\chi)$. We thus have to understand the connecting homomorphism δ. Observe that there is a canonical embedding $\iota : \mathbb{R}/\mathbb{Z} \to \mathrm{Homeo}^+(S^1)$ via the action by rotations, so we obtain a homomorphism $\widetilde{\chi} := \iota \circ \chi : G \to$

$\mathrm{Homeo}^+(S^1)$. Then a really unpleasant diagram chase shows that $\delta(\chi) = \widetilde{\chi}^* e_b$. We thus find

$$\rho^*(e_b) = \chi^*(e_b).$$

Now since $\mathbb{R}/\mathbb{Z}$ is abelian, the homorphism χ (and hence $\widetilde{\chi}$) vanishes on $[G,G]$. It follows that $\rho|_{[G,G]}^* e_b = 0$, whence $[G,G]$ has a fixed point. $\square$

4.3. **Surface groups, hyperbolizations and bounded fundamental class.** Let Σ_g be a closed oriented surface of genus g and $\Gamma_g := \pi_1(\Sigma_g)$. A representation $h : \Gamma_g \to \mathrm{PU}(1,1)$ will be called a *hyperbolization* if it is the holonomy representation of an oriented hyperbolic structure on Σ_g, and an anti-hyperbolization if it is the holonomy representation of a negatively-oriented hyperbolic structure on Σ_g. Here we think of $\mathrm{PU}(1,1)$ as the identity component of the automorphism group of the Poincaré disc $\mathbb{D}$. By classical results of Dehn, Nielsen and Baer, every discrete and faithful representation is either a hyperbolization or an anti-hyperbolization. Moreover, the unique outer automorphism of $\mathrm{PU}(1,1)$ swaps hyperbolizations and anti-hyperbolization. In studying discrete, faithful representations into $\mathrm{PU}(1,1)$ we can thus focus on hyperbolizations.

The action of $\mathrm{PU}(1,1)$ on $\mathbb{D}$ by fractional linear transformations extends continuously to the boundary $S^1 = \partial\mathbb{D}$. This boundary action gives rise to an embedding $\iota : \mathrm{PU}(1,1) \to \mathrm{Homeo}^+(S^1)$. Therefore, every hyperbolization $h : \Gamma_g \to \mathrm{PU}(1,1)$ gives rise to an embedding

$$\check{h} := \iota \circ h : \Gamma_g \to \mathrm{Homeo}^+(S^1),$$

which for lack of a better name we call a *hyperbolic boundary action*. The following is a weak version of hyperbolic stability:

Lemma 4.4. *Any two hyperbolic boundary actions $\check{h}_1, \check{h}_2 : \Gamma_g \to \mathrm{Homeo}^+(S^1)$ are conjugate.*

The strong version of hyperbolic stability says that they are actually conjugate by a Hölder continuous homomorphism, but we will not need this fact here. We recall that the automorphism group $\mathrm{Aut}(G)$ of a group G acts on the (bounded) cohomology ring by functoriality. It is a classical fact in group cohomology, that the action of the inner automorphism group on cohomology is trivial, and this fact (and its proof) carry over to bounded cohomology. We deduce:

Corollary 4.5. *Let $\alpha \in H^\bullet(\mathrm{Homeo}^+(S^1), \mathbb{R})$ or $\alpha \in H_b^\bullet(\mathrm{Homeo}^+(S^1), \mathbb{R})$ and let $\check{h} : \Gamma_g \to \mathrm{Homeo}^+(S^1)$ be a hyperbolic boundary action. Then the class $\alpha|_{\Gamma_g} := \check{h}^*\alpha$ is independent of the choice of hyperbolic boundary action.*

This allows us to define:

Definition 4.6. The *fundamental class*, respectively *bounded fundamental class* of Σ_g are defined as $\kappa^{\Sigma_g} := e^{\mathbb{R}}(S_1)|_{\Gamma_g} \in H^2(\Gamma_g)$, respectively $\kappa_b^{\Sigma_g} := e_b^{\mathbb{R}}(S_1)|_{\Gamma_g} \in H_b^2(\Gamma_g)$.

The name fundamental class will be justified below, when we relate it to the hyperbolic volume form. For the moment, it is just a name. The term bounded fundamental class was introduced in [8], where the importance of this class was first fully realized. Namely, it is indeed fundamental in the sense that it detects certain types of small subgroups:

Definition 4.7. A subgroup $H < \mathrm{PU}(1,1)$ is called *elementary* if it has a finite orbit in $\overline{\mathbb{D}}$.

Theorem 4.8 (Detecting elementary subgroups with bounded cohomology)**.** *Let $\Lambda < \Gamma_g$ be a subgroup of a surface group. Assume that $(\kappa_b^{\Sigma_g})|_\Lambda = 0 \in H_b^2(\Lambda;\mathbb{Z})$. Then the image of Λ under any hyperbolization is elementary. If Λ is normal in Γ_g, then Λ is trivial.*

Proof. Let us fix a hyperbolization $h : \Gamma \to \mathrm{PU}(1,1)$ and an associated boundary action $\widetilde{h} : \Gamma \to \mathrm{Homeo}^+(S^1)$. Denote by $\mathcal{F}$ the set of $\widetilde{h}([\Lambda,\Lambda])$-fixed points in S^1. Since $(\widetilde{h}|_\Lambda)^* e_b^{\mathbb{R}} = 0$ we deduce from Corollary 4.3 that $\mathcal{F}$ is non-empty. By construction, $\mathcal{F}$ is $\widetilde{h}(\Lambda)$-invariant, hence if $\mathcal{F}$ is finite then $h(\Lambda)$ is elementary. We may thus assume that $\mathcal{F}$ is infinite. However, the only subgroup of $\mathrm{PU}(1,1)$ with infinitely many fixed points on the circle is the trivial group. It follows that $\widetilde{h}([\Lambda,\Lambda]) = \{e\}$, whence h factors through the abelianization of Λ. Now every abelian subgroup of $\mathrm{PU}(1,1)$ is elementary. This proves the first statement, and the second statement follows from the fact that the hyperbolization of a surface group has no normal elementary subgroups. $\square$

Before we turn to applications we need one more technical tool:

Lemma 4.9. *Let $\check{h} : \Gamma_g \to \mathrm{Homeo}^+(S^1)$ be a boundary hyperbolization and $\widetilde{\check{h}} : \widetilde{\Gamma_g} \to \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$ be a lift. Then*

$$T(\widetilde{\check{h}}(\widetilde{\Gamma_g})) \subset \mathbb{Z}.$$

Proof. Every $g \in \check{h}(\Gamma_g)$ is hyperbolic, hence has a fixed point on S^1. Thus, in every fiber of $\widetilde{\check{h}}(\widetilde{\Gamma_g})$ over $\check{h}(\Gamma_g)$ there is an element g^* which fixes a point in $\mathbb{R}$. For this elements we have $T(g^*) = 0$ by the explicit formula for T. Now if h is a different element in the same fiber, then $h = g^*\tau$, where τ is an integer translation. Now translations are central, hence $T(h) = T(g^*) + T(\tau) = T(\tau) \in \mathbb{Z}$ by Corollary 3.13. $\square$

4.4. **A cohomological criterion for injectivity.** We now present a first application of bounded cohomology to representation theory of surface groups, based on Theorem 4.8. Concerning surface groups, we use the notation introduced in the previous section

Theorem 4.10 (Injectivity criterion)**.** *Let G be a group and $\alpha_b \in H_b^2(G;\mathbb{R})$. Assume that α_b admits a rational representative. If $\rho : \Gamma_g \to G$ is a representation with*

$$\rho^*\alpha_b = \lambda \cdot \kappa_b^{\Sigma_g}$$

for some $\lambda \neq 0$, then ρ is injective.

Proof. We represent α_b by a quasi-corner (f_G, p_G). Clearing denominators, we may assume without loss of generality that the class α_b actually has an integral representative. Then we can choose p_G so that $\ker(p_G) \cong \mathbb{Z}$. We also fix a hyperbolization

h and a corresponding boundary action $\check{h}$. Then both ρ and $\check{h}$ lift, giving rise to the diagram

$$\begin{tikzcd} \mathbb{R} & \widetilde{G} \arrow[l, "f_G"'] \arrow[d, "p_G"'] & \widetilde{\Gamma_g} \arrow[l, "\widetilde{\rho}"'] \arrow[r, "\widetilde{\check{h}}"] \arrow[d, "p_{\Gamma_g}"'] & \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) \arrow[r, "T"] \arrow[d, "p"] & \mathbb{R} \\ & G & \Gamma_g \arrow[l, "\rho"] \arrow[r, "\check{h}"'] & \mathrm{Homeo}^+(S^1). \end{tikzcd} \tag{4.2}$$

In view of Corollary 3.23 our assumption thus amount to the existence of a homomorphism $\psi : \widetilde{\Gamma_g} \to \mathbb{R}$ such that

$$f_G \circ \widetilde{\rho} = -\lambda \cdot (T \circ \widetilde{\check{h}}) + \psi. \tag{4.3}$$

Now consider the group $\Lambda := \ker(\rho)$ and let $\widetilde{\Lambda} := p_{\Gamma_g}^{-1}(\Lambda)$. We have

$$p_G(\widetilde{\rho}(\widetilde{\Lambda})) = \rho(p_{\Gamma_g}(\widetilde{\Lambda})) = \rho(\Lambda) = \{0\} \Rightarrow \widetilde{\rho}(\widetilde{\Lambda}) \subset \mathbb{Z}.$$

In particular, $\widetilde{\rho}(\Lambda)$ is abelian, and thus $(f_G \circ \widetilde{\rho})|_\Lambda$ is a homomorphism. It then follows from (4.3) that $T \circ \widetilde{\check{h}}|_{\widetilde{\Lambda}}$ is a homomorphism. This implies that

$$\kappa_b^{\Sigma_g}|_\Lambda = \check{h}^* e_b^{\mathbb{R}}(S^1) = p_{\Gamma_g}^{-1}[d(T \circ \widetilde{\check{h}}|_{\widetilde{\Lambda}})] = 0.$$

We thus deduce from Theorem 4.8 that $\Lambda = \{e\}$, whence ρ is injective. $\square$

4.5. **A cohomological criterion for discreteness.** A variation of the same ideas as in the proof of Theorem 4.10 can also be used to detect whether a representation $\rho : \Gamma_g \to G$ of a surface group into Lie group G has discrete image.[3]

In order to discuss discreteness of a representation, we clearly need to take the topology of G into account. The right way to do this is to develop a topological version of bounded cohomology; we will discuss this approach in some details below. However, for the present purpose we will choose an ad hoc approach which serves it purpose. Given a topological group G, let us call a quasi-corner

$$\begin{tikzcd} \widetilde{G} \arrow[r, "f"] \arrow[d, "p"] & \mathbb{R} \\ G & \end{tikzcd}$$

over G a *topological quasi-corner* is $\widetilde{G}$ is a topological group, p is a covering map and f is a continuous[4] homogeneous quasimorphism. Let us call a class $\alpha \in H_b^2(G; \mathbb{R})$ *continuous* if it can be represented by a topological quasi-corner. Then we have:

Theorem 4.11 (Discreteness criterion)**.** *Let G be a Lie group and $\alpha_b \in H_b^2(G; \mathbb{R})$ be continuous. Assume that α_b admits a rational representative. If $\rho : \Gamma_g \to G$ is a representation with*

$$\rho^* \alpha_b = \lambda \cdot \kappa_b^{\Sigma_g}$$

[3]Since the proof of the main theorem is rather technical and its ideas and techniques are not used in the sequel, this section can be skipped without loss of continuity. However, we will use the results later on.

[4]If G is assumed locally compact second countable then every Borel homogeneous quasimorphism is continuous, so the assumption can be weakened

for some $\lambda \in \mathbb{Q}^\times$, then (it is injective and) $\rho(\Gamma_g)$ is discrete.

Proof. Again we may after clearing denominators assume that α_b is integral and represented by a continuous quasi-corner $C(f_G, p_G)$ with $\ker(p) \cong \mathbb{Z}$. We may also assume that $f_G(\ker(p_G)) = \frac{1}{D} \cdot \mathbb{Z}$, where D is the denominator we cleared. As in the proof of Theorem 4.10 we then construct the diagram (4.2) and obtain the formula (4.3). Now define

$$L := \overline{\rho([\Gamma_g, \Gamma_g])}, \quad \widetilde{L} := p_G^{-1}(L).$$

Since p_G is a covering we have

$$\widetilde{L} := p_G^{-1}(\overline{\rho[\Gamma_g, \Gamma_g]}) = \overline{p_G^{-1}(\rho([\Gamma_g, \Gamma_g]))} = \overline{\widetilde{\rho}([\widetilde{\Gamma_g}, \widetilde{\Gamma_g}]) \cdot \ker(p_G)}.$$

We claim that $f_G(\widetilde{L}) \subset \mathbb{R}$ is discrete. Since f_G it is continuous, it suffices to show that in fact $f_G(\widetilde{\rho}([\widetilde{\Gamma_g}, \widetilde{\Gamma_g}]) \cdot \ker(p_G))$ is discrete. Now, since $\widetilde{\rho}([\widetilde{\Gamma_g}, \widetilde{\Gamma_g}]$ and $\ker(p_G)$ commute, it follows from Corollary 3.13 that

$$f_G(\widetilde{\rho}([\widetilde{\Gamma_g}, \widetilde{\Gamma_g}]) \cdot \ker(p_G)) = f_G(\widetilde{\rho}([\widetilde{\Gamma_g}, \widetilde{\Gamma_g}])) + f_G(\ker(p_G)).$$

Concerning the first summand, the identity (4.3) in combination with Lemma 4.9 yields

$$f_G(\widetilde{\rho}([\widetilde{\Gamma_g}, \widetilde{\Gamma_g}])) = -\lambda \cdot (T \circ \widetilde{\widetilde{h}})([\widetilde{\Gamma_g}, \widetilde{\Gamma_g}]) \subset \lambda \cdot \mathbb{Z},$$

since the homomorphism ψ vanishes on commutators. Since the second summand is contained by $\frac{1}{D}\mathbb{Z}$ by assumption and since λ was assumed rational, we deduce that $f_G(\widetilde{L}) \subset \mathbb{R}$ is indeed discrete.

Let now $\widetilde{L}^o$ denote the identity component of $\widetilde{L}$. Since $\widetilde{L}^o$ is connected and $f_G(\widetilde{L})$ is discrete we have $f_G(\widetilde{L}^o) \equiv 0$. Combining this with the observation that p_G is an open map and hence $p_G(\widetilde{L}^o) = L^o$ we deduce that

$$0 = [df_G|_{\widetilde{L}^o}] = p_G^* \alpha_b|_{\widetilde{L}^o} = (p_G|_{\widetilde{L}^o})^*(\alpha_b|_{L^o}),$$

hence $\alpha_b|_{L^o} = 0$.

Now consider $\Delta := \rho^{-1}(\rho[\Gamma_g, \Gamma_g] \cap L^o)$. We have

$$(\rho|_\Delta)^*(\alpha_b) = (\rho|_\Delta)^*(\alpha_b|_{L^o}) = 0,$$

hence $(\lambda \cdot \kappa_b^{\Sigma_g})|_\Delta = 0$ by assumption. Since $\lambda \neq 0$ we deduce that $\kappa_b^{\Sigma_g}|_\Delta = 0$ and thus $\Delta = 0$ by Theorem 4.8. Consequently, $\rho[\Gamma_g, \Gamma_g] \cap L^o = \{e\}$.

Up to now, everything works for arbitrary topological groups G. Now we use that G is a Lie group. Namely, the closed subgroup $L < G$ is a Lie group, whence L^o is open in L. It follows that $\rho([\Gamma_g, \Gamma_g]) \cap L^o$ is dense in L^o, hence $L^o = \{e\}$. It follows that L, and consequently $\rho([\Gamma_g, \Gamma_g])$ is discrete.

Now $\rho(\Gamma)$ normalizes $\rho([\Gamma_g, \Gamma_g])$, hence $\overline{\rho(\Gamma)}$ normalizes $\overline{\rho([\Gamma_g, \Gamma_g])} = \rho([\Gamma_g, \Gamma_g])$. Consequently, $\rho([\Gamma_g, \Gamma_g])$ centralizes the identity component $\overline{\rho(\Gamma_g)}^o$. Now $\rho(\Gamma_g)$ is dense in $\overline{\rho(\Gamma_g)}$ and $\overline{\rho(\Gamma_g)}^o$ is open. Thus, if we assume $\overline{\rho(\Gamma_g)}^o \neq \{e\}$, then we can find $\gamma \in \Gamma_g$ with $\rho(\gamma) \in \overline{\rho(\Gamma_g)}^o \setminus \{e\}$. We then deduce that $\rho(\gamma)$ centralizes $\rho([\Gamma_g, \Gamma_g])$.

Now apply Theorem 4.10 and deduce that ρ is injectivity. It follows that γ centralizes $[\Gamma_g, \Gamma_g]$. However, the centralizer of $[\Gamma_g, \Gamma_g]$ in Γ_g is trivial. This contradiction show that $\overline{\rho(\Gamma_g)}^o = \{e\}$, whence $\rho(\Gamma_g)$ is discrete. $\square$

In the proof we clearly use that $\lambda \in \mathbb{Q}^\times$ and that G is a Lie group. It is unclear to us, whether these assumptions are necessary. Rationality of λ is used in an essential way, whereas the Lie group property of G seems less essential (and is unfortunately very restrictive in terms of applications). In any case, at least for representations of surface groups into Lie groups we have found a cohomologival criterion which guarantees that the representation in question is discrete and faithful.

On first sight our criterion might not look very useful, since it basically replaces a mysterious condition by an even more mysterious one. The point we will have to make below is that in many interesting geometric situations one can actually check the cohomological condition. This will in particular be the case for maximal and Shilov-Anosov representations. Before we can make the link, however, we need to study some more bounded cohomology.

5. Bounded cohomology II: Bounded Kähler classes and boundary resolutions

5.1. **Continuous bounded cohomology of Lie groups.** There is a version of bounded cohomology for topological groups, due to Burger and Monod [9, 10, 11, 29]. Given a topological group G one can replace in the definition of bounded cohomology the space $l^\infty(G^{n+1};\mathbb{R})$ of bounded functions by the space of *continuous bounded functions* $C_b(G^{n+1};\mathbb{R})$. The resulting cohomology is called the *continuous bounded cohomology* of G and denoted $H^\bullet_{cb}(G;\mathbb{R})$. As for bounded cohomology one shows $H^0_{cb}(G;\mathbb{R}) \cong \mathbb{R}$ and $H^1_{cb}(G;\mathbb{R}) = \{0\}$. Thus the first interesting continuous bounded cohomology is the second one.

What is the second continuous bounded cohomology of a Lie group? We first recall that every Lie group G is of a semidirect product $G = R_uG \rtimes G_{ss}$, where R_uG is solvable and G_{ss} is semisimple with trivial center, hence decomposes as a product of simple factors. Now G_{ss} decomposes further as $G_{ss} = G^c_{ss} \times G^{nc}_{ss}$, where G^c_{ss} is compact and all simple factors of G^{nc}_{ss} are non-compact.

Proposition 5.1. $H^2_{cb}(G;\mathbb{R}) = H^2_{cb}(G^{nc}_{ss})$.

Proof. The 5-term sequence which we used in the proof of Proposition 3.19 exists also in continuous bounded cohomology. Thus, taking the quotient of a group by an amenable subgroup does not change the second bounded cohomology. Now R_uG is solvable and G^c_{ss} is compact, hence both are amenable. □

The result holds actually in arbitrary degrees and can be proved directly by using averaging operators constructing the means of R_uG and G^c_{ss}. We also mention the following result of Monod, which is special to degree 2, and can also be deduced from the five-term sequence:

Lemma 5.2. $H^2_{cb}(G_1 \times G_2;\mathbb{R}) \cong H^2_{cb}(G_1;\mathbb{R}) \oplus H^2_{cb}(G_2;\mathbb{R})$.

Combining this with the proposition we see that in our study of $H^2_{cb}(G;\mathbb{R})$ we can restrict attention to non-compact simple Lie groups. As in the case of bounded cohomology we have a comparison map $H^2_{cb}(G;\mathbb{R}) \to H^2_c(G;\mathbb{R})$, where the right hand side denotes cohomology with continuous cochains. Now we have:

Proposition 5.3. *For every semisimple non-compact Lie group G the degree 2 comparison map $H^2_{cb}(G;\mathbb{R}) \to H^2_c(G;\mathbb{R})$ is injective.*

Proof. Using Monod's lemma one immediately reduces to the simple case. As in the case of usual bounded cohomology one shows that the kernel of the comparison map is given by by continuous homogeneous quasimorphisms modulo homomorphisms. Thus the proposition amounts to showing that every continuous homogeneous quasimorphism on a simple Lie group is trivial. This follows from a general property of simple Lie groups called *bounded generation by unipotents.* We give the details for $G = \mathrm{SL}_n(\mathbb{R})$. Let $g_{ij}(t) := \exp(tE_{ij})$, where E_{ij} is the elementary matrix which is 1 at the ij-th entry and 0 everywhere else. Then for $i \neq j$

$$\lim_{s\to\infty} g_{ii}(s)g_{ij}(t)g_{ii}(s)^{-1} = e,$$

hence for every continuous homogeneous quasimorphism f and $i \neq j$ we have by Proposition 3.14

$$f(g_{ij}(t)) = \lim_{s\to\infty} f(g_{ij}(t)) = \lim_{s\to\infty} f(g_{ii}(s)g_{ij}(t)g_{ii}(s)^{-1}) = 0.$$

Now, by the Gauss algorithm, every matrix in $SL_n(\mathbb{R})$ can be written as a product of no more than $100n^2$ matrixes of the form $g_{ij}(t)$ with $i \neq j$. Therefore, f is bounded by $100n^2 \cdot D(f)$. However, a bounded homogeneous quasimorphism is trivial. $\square$

So in order to understand $H^2_{cb}(G;\mathbb{R})$ for Lie groups, it suffices to understand $H^2_c(G;\mathbb{R})$ for simple Lie groups.

5.2. **Bounded Kähler class and Hermitian Lie groups.**

Example 5.4. Consider the action of $G := \mathrm{PU}(1,1)$ on the Poincaré disc $\mathbb{D}$ and denote by $\omega \in \Omega^2(\mathbb{D})^G$ the hyperbolic volume form $\omega = \frac{1}{1-(x^2+y^2)^2}dxdy$. Given $(x_0, x_1, x_2) \in \mathbb{D}$ denote by $\Delta(x_0, x_1, x_2)$ the geodesic triangle with corners x_0, x_1, x_2 and fix a basepoint $o \in \mathbb{D}$. Then we obtain a homogeneous 2-cocycle $c_{\omega,o} : G^3 \to \mathbb{R}$ given by

$$c_{\omega,o}(g_0, g_1, g_2) := \frac{1}{2\pi}\int_{\Delta(g_0.o,g_1.o,g_2.o)} \omega.$$

Exercise 5.5. Show that the cohomology class $\kappa_G := [c_{\omega,o}] \in H^2_c(G,\mathbb{R})$ is independent of the choice of basepoint o. [Hint: Use that ω is closed.]

We refer to κ_G as the *Kähler class* of G. It turns out that

$$H^2_c(G,\mathbb{R}) \cong \mathbb{R}\cdot\kappa_G.$$

Since every geodesic triangle is contained in an ideal triangle (of volume π) the cocycle $c_{\omega,o}$ is bounded by $1/2$. It thus defined a class $\kappa^b_G := [c_{\omega,o}] \in H^2_c(G,\mathbb{R})$, called the *bounded Käbler class.* By injectivity of the comparison map, it is the unique pre-image of the Kähler class in bounded cohomology.

Example 5.4 is the prototype of a very general theory based on the following observation of Cartan:

Lemma 5.6 (E. Cartan)**.** *Let G be a simple Lie group, $\mathcal{X} = G/K$ the associated symmetric space and $\omega \in \Omega^n(\mathcal{X})^G$ a G-invariant form. Then ω is closed.*

Proof. Let $o \in \mathcal{X}$ be a basepoint and assume $\omega \in \Omega^n(\mathcal{X})^G$. Since the geodesic reflection σ_o at o normalizes G we have $\sigma_o^*\omega \in \Omega^n(\mathcal{X})^G$. Now, $\sigma_o^*\omega = (-1)^n\omega$; indeed, the relation holds at the basepoint o and both sides are G-invariant. We can apply the same argument also to $d\omega$ to obtain $\sigma_o^*(d\omega) = (-1)^{n+1}d\omega$. Now we get

$$(-1)^n d\omega = d((-1)^n\omega) = d\sigma_o^*\omega = \sigma_o^*(d\omega) = (-1)^{n+1}d\omega,$$

showing that $d\omega = 0$. □

By Cartan's lemma, we can integrate invariant forms on $\mathcal{X}$ over geodesic simplices and thereby construct a map

$$I : \Omega^n(\mathcal{X})^G \to H_c^n(G;\mathbb{R}), \quad I(\omega) = \left[(g_0, \ldots, g_n) \mapsto \frac{1}{2\pi}\int_{\Delta(g_0,\ldots,g_n)} \omega\right].$$

Lemma 5.7 (van Est)**.** *The map $I : \Omega^n(\mathcal{X})^G \to H_c^n(G;\mathbb{R})$ is an isomorphism.*

Proof. $\Omega^n(X)$ is an injective resolution of the trivial module; by Cartan's lemma the differentials in the complex $(\Omega^n(X)^G, d)$ are trivial. □

One can try to estimate the integrals appearing in the explicit van Est formula. In degree 2 (and only in degree 2), J. L. Dupont succeeded in doing so. Thereby he proved:

Proposition 5.8 (Dupont, [18])**.** *The cocycles in the image of the degree* 2 *van Est map are bounded.*

This yields in particular surjectivity of the comparison map in degree 2. For the state of the art in higher degree see [24]. Combining everything we said so far we obtain:

Corollary 5.9. *Let G be a simple Lie group with symmetric space $\mathcal{X} = G/K$. Then*

$$H_{cb}^2(G;\mathbb{R}) \cong H_c^2(G;\mathbb{R}) \cong \Omega^2(\mathcal{X})^G.$$

Definition 5.10. A simple Lie group is called *Hermitian* if $\Omega^2(\mathcal{X})^G \neq \{0\}$.

The geometry of Hermitian Lie groups is well-known. We only provide a few key-words:

- If G is a Hermitian simple Lie group, then $\dim \Omega^2(\mathcal{X})^G = 1$.
- Every $\omega \in \Omega^2(\mathcal{X})^G \setminus \{0\}$ is in fact a Käbler form on $\mathcal{X}$, i.e. there exists a complex structure $J : T\mathcal{X} \to T\mathcal{X}$ such that $g(v,w) := \omega(v, Jw)$ is a Riemannian metric. This complex structure is unique, so $\mathcal{X}$ is canonically a complex manifold. We refer to this complex manifold as a *Hermitian symmetric space.* We can normalize ω so that the minimal holomorphic sectional curvature of the Hermitian metric $H = g + i\omega$ is -1. With this normalization, ω becomes unique. We then denote by κ^G and κ_b^G the corresponding classes in $H_c^2(G;\mathbb{R})$ and $H_{cb}^2(G;\mathbb{R})$, called the *Kähler class*, respectively *bounded Käbler class* of G.

- Via the Harish-Chandra embedding theorem, every Hermitian symmetric space $\mathcal{X}$ is isomorphic as a complex manifold to a bounded domain $\mathcal{D} \subset \mathbb{C}^{\dim \mathcal{X}}$, which is unique up to unique isomorphism. The Riemannian metric g (and hence ω) can be recovered from $\mathcal{D}$ as the Bergman metric associated with the embedding into $\mathbb{C}^{\dim \mathcal{X}}$. A symmetric domain $\mathcal{D}$ isomorphic to a symmetric space is called a *bounded symmetric domain.* Thus Hermitian Lie groups are precisely the identity components of automorphism groups (i.e. groups of biholomorphisms) of bounded symmetric domains.
- A tube domain in $\mathbb{C}^N$ is a subdomain $T = \mathbb{R}^N + i\Omega$, where $\Omega \subset \mathbb{R}^N$ is an open convex sone. A bounded symmetric domain is *of tube type* it is biholomorphic to a tube domain. The prototype of a tube domain is the upper half plane; in particular, $\mathbb{D}$ is of tube type. We say that a simple Hermitian Lie group is of tube type if the associated bounded symmetric domain has this property. Thus $\mathrm{PU}(1,1)$ is of tube type.
- Other example of Hermitian Lie groups of tube type include the real-split symplectic groups $\mathrm{Sp}(2n)$ and the groups $SU(n,n)$. The groups $SU(p,q)$ for $p \neq q$ are examples of Hermitian Lie groups, which are *not* of tube type.

All the cohomological methods we have presented above use the second bounded cohomology. Thus, with these methods, we can only obtain results about representations into Hermitian Lie groups. For non-Hermitian Lie groups like $\mathrm{SL}_n(\mathbb{R})$ these methods are completely useless. In order to treat such groups, it would be necessary to obtain a better understanding of bounded cohomology in degrees > 2.

5.3. **The boundary resolution.** Consider again the example of $PU(1,1)$ acting on the Poincaré disc $(\mathbb{D}, \omega)$. We have constructed a family of (boundedly) cohomologous cocycles $c_{\omega,o}$ depending on some basepoint $o \in \mathbb{D}$. What happens if o wanders off to the boundary?

Recall that every triple $(\xi_0, \xi_1, \xi_2) \in (S^1)^3 = (\partial\mathbb{D})^3$ defines an ideal triangle $\Delta(\xi_0, \xi_1, \xi_2)$. Thus for $\xi \in S^1$ we can define a cocycle $c_{\omega,\xi}$ by the same formula

$$c_{\omega,\xi}(g_0, g_1, g_2) := \frac{1}{2\pi} \int_{\Delta(g_0.\xi, g_1.\xi, g_2.\xi)} \omega.$$

as before. However, something strange happens when passing to the boundary: Every ideal simplex has volume π! Thus, $c_{\omega,\xi}(g_0, g_1, g_2) = \frac{1}{2} \cdot \mathrm{o}(g_0.\xi, g_1.\xi, g_2.\xi)$, where

$$\mathrm{o}(\xi_0, \xi_1, \xi_2) = \begin{cases} +1, & (\xi_0, \xi_1, \xi_2) \text{ non-degenerate and positively oriented} \\ -1 & (\xi_0, \xi_1, \xi_2) \text{ non-degenerate and negatively oriented} \\ 0 & (\xi_0, \xi_1, \xi_2) \text{ degenerate.} \end{cases}$$

is the orientation cocycle of the circle. This cocycle is still Borel measurable and bounded, but no longer smooth. Now, by a theorem of D. Wigner, the cohomology of a Lie group computed from Borel measurable cochains is isomorphic to the cohomology computed from continuous cochains, in symbols

$$H^\bullet_B(G; \mathbb{R}) \cong H^\bullet_c(B; \mathbb{R}).$$

Under this isomorphism, the cocycles $c_{\omega,\xi}$ represent the Kähler class. Burger and Monod have proved that there is a similar isomorphism

$$H^\bullet_{Bb}(G; \mathbb{R}) \cong H^\bullet_{cb}(B; \mathbb{R}),$$

sending the class of $c_{\omega,\xi}$ to the bounded Kähler class. This is part of a more general theory, which we briefly sketch:

The space S^1 is the prototype of a *boundary* for the group $\mathrm{PU}(1,1)$. For the purposes of these notes let us call a standard Borel G-space B a *weak G-boundary* if it an amenable G-space in the sense of Zimmer and the action of G on $B \times B$ is ergodic. If you do not know what these words mean think of the following examples, which suffice for our purposes: If G is a simple Lie group and P is a minimal parabolic, then G/P is a weak G-boundary. Similarly, if $\Gamma < G$ is a lattice (e.g. a surface group in $\mathrm{PU}(1,1)$), then G/P is still a weak Γ-boundary. For Gromov-hyperbolic groups, the Gromov boundary is a weak boundary. With this terminology understood, the starting point of the Burger-Monod approach to (continuous) bounded cohomology can be formulated as follows:

Proposition 5.11 (Boundary resolution)**.** *Let G be a locally-compact, 2nd countable group and B a weak G-boundary. Then*

$$0 \to \mathbb{R} \to L^\infty(B) \to L^\infty(B^2) \to L^\infty(B^3) \to \dots,$$

where all maps are given by the usual homogeneous differentials, is an injective augmented resolution of the trivial Banach G-module $\mathbb{R}$. Thus there is a canonical isomorphism

$$H^\bullet_{cb}(G;\mathbb{R}) \cong H^\bullet(0 \to L^\infty(B)^G \to L^\infty(B^2)^G \to L^\infty(B^3)^G \to \dots)$$

Exercise 5.12. Show that the boundary resolution is homotopic to the *alternating boundary resolution*

$$0 \to L^\infty_{alt}(B) \to L^\infty_{alt}(B^2) \to L^\infty_{alt}(B^3) \to \dots,$$

where $L^\infty_{alt}(B^n) \subset L^\infty(B^n)$ denotes the subspace of alternating functions.

Let us specialize to degree 2: We have

$$H^2_{cb}(G;\mathbb{R}) = \frac{\mathcal{Z}^2_{alt}(B)^G}{\mathcal{B}^2_{alt}(B)^G} := \frac{\ker(d : L^\infty_{alt}(B^3)^G \to L^\infty_{alt}(B^4)^G)}{\mathrm{Im}(d : L^\infty_{alt}(B^2)^G \to L^\infty_{alt}(B^3)^G)}.$$

Now, since G acts ergodically on B^2, every G-invariant function on B^2 is constant, hence every alternating G-invariant function on B^2 is 0. Thus there are no coboundaries in degree 2 and we have:

Proposition 5.13. $H^2_{cb}(G;\mathbb{R}) = \mathcal{Z}^2_{alt}(B)^G = \{c \in L^\infty_{alt}(B^3)^G \,|\, dc = 0\}$.

This is the reason why the boundary resolution is very convenient for computations in degree 2.

5.4. **Functoriality and transfer.** There is one major problem with the boundary resolution, which concerns functoriality. Assume we are given two Lie groups G_1 and G_2 with respective weak boundaries B_1 and B_2 and a cohomology class $\alpha \in H^2_{cb}(G_2;\mathbb{R})$. How to pull back α via a homomorphism $\rho : G_1 \to G_2$. It is not completely trivial that there exists a unique equivariant map $\varphi : B_1 \to B_2$, although this is in fact the case by a result of Furstenberg (uniqueness is up to measure zero). However, even in the most favorable situations, the image of B_1 in B_2 will typically be a null set (unless B_1 and B_2 have the same dimension). However, it is not possible to restrict an L^∞-function (which is actually a function class, of course) to a null set. Therefore we cannot implement the pullback via φ. There is no solution to this problem, only work-arounds. The most useful one is due to Burger and Iozzi [4]:

Lemma 5.14 (Functoriality redeemed)**.** *Let G_1, G_2 be locally compact groups with respective weak boundaries B_1, B_2. Let $c : B_2^3 \to \mathbb{R}$ be a bounded, alternating, G-invariant measurable function with $dc(\xi_0, \ldots, \xi_3) = 0$ for all (not just almost all!) triples $(\xi_0, \ldots, \xi_3) \in B^3$. Then c defines a function class in $\mathcal{Z}^2_{alt}(B)^G$, which in turn defines a class $\alpha \in H^2_{cb}(G_2; \mathbb{R})$. Let $\rho : G_1 \to G_2$ be a continuous homomorphism and $\varphi : B_1 \to B_2$ be an equivariant measurable map. Then the class defined by $\varphi^* c$ in $H^2_{cb}(G_1; \mathbb{R})$ coincides with $\rho^*\alpha$.*

Note that in general there is no guarantee that a given cocycle can be represented by a function c as above. Fortunately, in all known examples of bounded cohomology classes on a boundary there is such a representative. (There are however examples, due to Bucher and Monod, of L^∞ cocycles on a flag variety associated with a non-minimal parabolic, which cannot be represented by a measurable function as above on that specific flag variety. This does not exclude the possibility that every boundary cocycle has a representative as above, but it shows that if there is a general principle, then it is necessarily quite subtle.)

One situation, where the above problems do not occur is when $\rho : \Gamma \to G$ is the inclusion of a lattice. In this case, ρ^* is realized by the inclusion map

$$\mathcal{Z}^2_{alt}(B)^G \to \mathcal{Z}^2_{alt}(B)^\Gamma.$$

In particular, the restriction map $H^2_{cb}(G; \mathbb{R}) \to H^2_b(\Gamma; \mathbb{R})$ is injective. The map $T^2_b : Z^2_{alt}(B)^\Gamma \to \mathcal{Z}^2_{alt}(B)^G$ given by

$$T^2_b f(x, y, z) := \int_{\Gamma\backslash G} f(gx, gy, gz) dg$$

yields an explicit left-inverse of this restriction map. By abuse of notation we also write T^2_b for the induced map $T^2_b : H^2_b(\Gamma; \mathbb{R}) \to H^2_{cb}(G; \mathbb{R})$, which is called the *bounded transfer map* in degree 2.

5.5. **The generalized Maslov index and the tube type dichotomy.** Let G be a Hermitian simple Lie group of tube type and let $\kappa^G_{(b)}$ denote its (bounded) Kähler class. We know from the last section that there exists a unique bounded alternating cocycle $c : (G/P)^3 \to \mathbb{R}$ representing κ^G_b, where P is a minimal parabolic of G. The G-equivariant quotients of G/P are precisely the flag varieties G/Q, where Q is a parabolic containing P. Given such a parabolic Q there may or may not exist a cocycle $\bar{c} : (G/Q)^3 \to \mathbb{R}$ representing κ^G_b. If it does exists, then it is clearly unique (since already c is). In this case we say that κ^G_b can be represented on G/Q. It turns out that bounded Kähler classes can be represented on a flag variety associated with a distinguished maximal parabolic subgroup. We are now going to describe this distinguished flag variety geometrically.

As mentioned earlier, we can always realize G as the groups of biholomorphisms of a bounded symmetric domain $\mathcal{D} \subset \mathbb{C}^N$. An example to keep in mind is given by the matrix ball

$$\mathcal{D}_n := \{X \in M_n(\mathbb{C}) \,|\, M \text{ symmetric}, \mathbf{1} - M^*M \text{ positive definite}\},$$

for which we have $\mathrm{Aut}(\mathcal{D}_n) \cong \mathrm{PSp}(2n)$, the action being by generalized fractional linear transformations.

Exercise 5.15. Show that the matrix ball $\mathcal{D}_n$ is of tube type by providing an explicit biholomorphism with the Siegel upper half plane

$$\mathcal{H}_n := \{X + iY \in \mathrm{Sym}_n(\mathbb{C}) \,|\, X, Y \in \mathrm{Sym}_n(\mathbb{R}), Y \text{ positive definite}\}.$$

[Hint: For $n = 1$ the biholomorphism is the Cayley transform. Now observe that the explicit formula for the Cayley transform makes sense also for matrices.]

We now return to the general case: The topological boundary $\partial\mathcal{D} := \overline{\mathcal{D}} \setminus \mathcal{D}$ depends on the concrete realization $\mathcal{D}$. However, there is always a unique closed G-orbit in $\mathcal{D}$, which is independent of the realization and called the *Shilov boundary* $\check{S}$ of G. It is the smallest compact subset of $\partial\mathcal{D}$ such that the maximum principle holds, i.e.

$$\sup_{z \in \mathcal{D}} f(z) = \sup_{z \in \check{S}} f(z).$$

for every continuous function f on $\overline{\mathcal{D}}$, which is holomorphic in $\mathcal{D}$.

Example 5.16. In the case of the matrix ball, the topological boundary is formed by all complex symmetric matrices M for which $\mathbf{1} - M^*M$ is positive semi-definite, but not positive definite. The G-orbits inside this topological boundary are given by those matrices of a constant rank r with $0 \leq r \leq n-1$, i.e. there are n different G-orbits. The unique closed G-orbit, i.e. the Shilov boundary, is simply given by the rank 0 component

$$\check{S}_n := \mathrm{Sym}_n(\mathbb{C}) \cap U(n).$$

Returning to the general case we observe that since $\check{S}$ is closed in $\overline{\mathcal{D}}$ it is always a compact homogeneous G-space, i.e. a flag variety. Thus there exists a maximal parabolic Q (unique up to conjugation) such that

$$\check{S} \cong G/Q.$$

We refer to Q and its conjugates as *Shilov parabolics*. One can now try to proceed as in the $\mathrm{PU}(1,1)$-case and to push the Kähler cocycle $c_{\omega,o}$ to the Shilov boundary by letting $o \to \xi \in \check{S}$. It is not at all obvious that such an approach can work, and indeed one has to be very careful from which directions one approaches the Shilov boundary, in particularly for non-transversal triples of points. In any case, it is possible to construct a limiting cocycle $\beta_{\check{S}}$. For $\mathrm{PU}(1,1)$ we have seen that $\beta_{\check{S}}$ is a multiple of the orientation cocycle. In the case of the symplectic groups $\mathrm{Sp}(2n)$, the cocycle $\beta_{\check{S}}$ is also a classical object of symplectic geometry, namely the so-called *Maslov index*. For general Hermitian G, the *generalized Maslov index* $\beta_{\check{S}}$ was constructed by Clerc [13]. Once the generalized Maslov index is constructed, it is not too hard to see that it does realize the bounded Kähler class:

Proposition 5.17 (Shilov realizability, [13])**.** *Let G be a simple Lie group of Hermitian type, $\check{S}$ the associated Shilov boundary and κ_b^G the bounded Kähler class.*

(i) *The generalized Maslov index $\beta_{\check{S}}$ is a bounded measurable, alternating G-invariant function.*
(ii) *The generalized Maslov index is continuous on the set $\check{S}^{(3)}$ of pairwise transverse triples and satisfies $d\beta_{\check{S}} = 0$ pointwise.*
(ii) *The bounded Kähler class κ_b^G can be represented on $\check{S}$, and the unique representing cocycle is precisely the generalized Maslov index.*

We emphasize that in order to know the bounded cohomology class defined by the Maslov index it is enough to know $\beta_{\check{S}}$ on the generic set $\check{S}^{(3)}$. However, in order to use Lemma 5.14 one needs to know that $\beta_{\check{S}}$ extends to a pointwise measurable cocycle on all of $\check{S}^3$. This is the most difficult part of Proposition 5.17. On the other hand, the restriction of $\beta_{\check{S}}$ to the generic set $\check{S}^{(3)}$ is very-well understood. It was first described by Clerc and Ørsted in [14]. In [6] Burger, Iozzi and Wienhard provide a formula in terms of what they call the Hermitian triple product. A consequence of this formula is the following dichotomy:

Proposition 5.18 (Bounded Kähler class dichotomy, [6])**.**
(i) *If G is of tube type, then $\beta_{\check{S}}|_{\check{S}^{(3)}}$ is locally constant and thus takes only a finite set of rational values.*
(ii) *If G is not of tube type, then $\beta_{\check{S}}|_{\check{S}^{(3)}}$ takes a continuum of values.*

5.6. **Relation to bounded Euler class.** As pointed out by Iozzi in [25] one can use the boundary resolution to give a simple proof of the following fact, which will be needed below:

Proposition 5.19. *Let $\iota : \mathrm{PU}(1,1) \to \mathrm{Homeo}^+(S^1)$ denote the standard embedding. Then $\iota^* e_b^{\mathbb{R}} = \kappa_b^{\mathrm{PU}(1,1)}$.*

Proof. Let $H := \mathrm{Homeo}^+(S^1)$. Let $c : H^3 \to \mathbb{R}$ be the cocycle given by $c(h_0, h_1, h_2) = o(h_0.1, h_1.1, h_2.1)$. In view of the boundary model it then suffices to show that c is boundedly cohomologous to the standard representative e_σ of $e_b^{\mathbb{R}}$. In fact we claim that

$$e_\sigma - c = d\beta, \tag{5.1}$$

where

$$\beta(f, g) = \begin{cases} -\frac{1}{2} & f(1) \neq 1, \\ 0 & \text{otherwise.} \end{cases}$$

□

Exercise 5.20. Show the equality (5.1). (If you get stuck, you may want to consult [25]).

Note that as a consequence we also get $\iota^* e^{\mathbb{R}} = \kappa^{\mathrm{PU}(1,1)}$ for free, using naturality of the comparison map.

6. Applications to Shilov-Anosov representations

6.1. **Shilov-Anosov representations and bounded Kähler class.** We pause for a moment to give a first application of the material of the last section. Let G be a Hermitian Lie group of tube type, Q a Shilov-parabolic and Q^- a parabolic opposite G. It turns out that Q and Q^- are conjugate, hence $G/Q \cong G/Q^- \cong \check{S}$. Assume now that $\rho : \Gamma_g \to G$ is a (Q, Q^-)-Anosov-representation, or Shilov-Anosov representation for short. Using our machinery we prove:

Proposition 6.1. *Let $\rho : \Gamma_g \to G$ be a Shilov-Anosov representation. Assume that G is of tube type. Then there exists $\lambda \in \mathbb{Q}$ such that $\rho^* \kappa_b^G = \lambda \cdot \kappa_b^{\Sigma_g}$.*

Proof. By the Guichard-Wienhard characterization of Anosov representations (see [17]) there exists a continuous ρ-equivariant map $\varphi : S^1 \to \check{S}$ which maps $(S^1)^{(3)}$ to $\check{S}^{(3)}$. By Lemma 5.14 and (the hard part of) Proposition 5.17 the class $\rho^*\kappa_b^G$ is represented by the cocycle $\varphi^*\beta_{\check{S}}$. Now

$$\varphi^*\beta_{\check{S}}|_{(S^1)^{(3)}} \to \mathbb{Q}$$

is continuous with discrete image (by the bounded Kähler class dichotomy), thus locally constant. However, the only alternating cocycles on S^1, which are locally constant on $(S^1)^{(3)}$ are the multiples of the orientation cocycle. We deduce that $\varphi^*\beta_{\check{S}} = \lambda \cdot o$, and since the left-hand side is rational almost everywhere, so is the right hand side. □

6.2. **The Toledo invariant.** We would like to combine Proposition 6.1 and Theorem 4.11 to obtain discreteness and faithfulness of Shilov-Anosov representations. However, for this we need to know that the constant λ appearing in Proposition 6.1 is non-zero. We thus need to understand the meaning of this constant.

There is a special relevance to classes in $H^2(\Gamma_g;\mathbb{R})$. due to the fact that surfaces are two-dimensional. Namely, $H_2(\Sigma_g;\mathbb{R}) = \mathbb{R}\cdot[\Sigma_g]$, where $[\Sigma_g]$ is the fundamental class. Thus we have a canonical evaluation map

$$H^2(\Gamma_g;\mathbb{R}) \cong H^2(\Sigma_g;\mathbb{R}) \to \mathbb{R}, \quad \alpha \mapsto \langle \alpha, [\Sigma_g]\rangle.$$

Consider now the special case of κ^{Σ_g}. By Proposition 5.19 we have $\kappa^{\Sigma_g} = \kappa^{\mathrm{PU}(1,1)}|_{\Gamma_g}$; thus in the de Rham model, κ^{Σ_g} is represented by $\frac{1}{2\pi}$ times the hyperbolic volume form. We deduce that

$$\langle \kappa^{\Sigma_g}, [\Sigma_g]\rangle = \frac{1}{2\pi}\int_{\Sigma_g} d\mathrm{vol} = |\chi(\Sigma_g)| = 2-2g.$$

We conclude that if $\rho : \Gamma_g \to G$ is a representation with

$$\rho^*\kappa_b^G = \lambda\cdot\kappa_b^{\Sigma_g}, \tag{6.1}$$

then

$$\rho^*\kappa^G = \lambda\cdot\kappa^{\Sigma_g},$$

and hence

$$\langle \rho^*\kappa^G, [\Sigma_g]\rangle = \lambda\cdot\langle\kappa^{\Sigma_g}, [\Sigma_g]\rangle = \lambda\cdot|\chi(\Sigma_g)|,$$

i.e.

$$\lambda = \frac{\langle \rho^*\kappa^G, [\Sigma_g]\rangle}{|\chi(\Sigma_g)|}. \tag{6.2}$$

Definition 6.2. Let G be a Hermitian simple group, κ^G its Kähler class, Γ_g a surface group and $\rho : \Gamma_g \to G$ a representation. Then the number

$$T(\rho) := \langle \rho^*\kappa^G, [\Sigma_g]\rangle \in \mathbb{Q}$$

is called the *Toledo invariant* of the representation ρ.

Note that $T(\rho)$ takes a discrete set of rational values, since κ^G is rational. Moreover, as a consequence of (6.2) we see that λ as given in (6.1) vanishes if and only if the Toledo invariant vanishes. Combining this with Proposition 6.1 and Theorem 4.11 we have obtained a complete proof of the following result:

Theorem 6.3. *Let $\rho : \Gamma_g \to G$ be a Shilov-Anosov representation into a Hermitian Lie group of tube type with non-zero Toledo invariant. Then ρ is faithful and its image is discrete.*

Of course, as we have already seen in [17, 31], much more is true: Any Anosov representation is essentially discrete and faithful. However, the proof given here is still interesting in its own right, since one can push its idea quite a bit further: Firstly, the assumption that G be of tube type is actually unnecessary, since one can show that every Shilov-Anosov representation with non-zero Toledo invariant preserves a subdomain of tube type. Secondly, and more importantly, we manage in [1] to extend the result to cover also the following case:

Theorem 6.4. *Let G be a Hermitian Lie group and $\rho : \Gamma_g \to G$ a representation of non-zero Toledo invariant which is a limit of Shilov-Anosov representations. Then ρ is faithful and its image is discrete.*

Using the theorem we provide new examples of discrete, faithful representations which are not Anosov. The following problem is open to the best of my knowledge: Consider a discrete and faithful representation $\rho : \Gamma_g \to \mathrm{Sp}(4)$, which is Zariski-dense and of non-zero Toledo invariant. Is it true that (6.1) holds for some $\lambda \in \mathbb{R}$? Representations satisfying this condition are called *weakly maximal*. So far we do not know any other discrete faithful representations, but this is probably mainly because we have no other tools to detect discreteness and faithfulness than the cohomological ones explained here.

7. Bounded cohomology III: The Gromov seminorm

7.1. **Motivation: Boundedness of the Toledo-invariant.** We have seen in the last section that the Toledo-invariant of a representation of a surface group into a Hermitian simple Lie group yields valuable information. We observe:

Proposition 7.1. *For a given Hermitian simple Lie group G, the Toledo invariant $T : \mathrm{Rep}(\Gamma_g; G) \to \mathbb{R}$ takes only finitely many values.*

Proof. It is not hard to see that T is continuous; since it range is discrete, it is thus locally constant. However, $\mathrm{Rep}(\Gamma_g; G)$ as an algebraic variety has only finitely many connected components. □

We thus know that the supremum $C_{G,\Gamma_g} := \sup_\rho |T(\rho)|$ taken over the representation variety $\mathrm{Rep}(\Gamma_g; G)$ is finite - can we determine its value? Since the range of the Toledo invariant is finite, the supremum is actually attained - but for which representations? We will answer these type of questions using bounded cohomology.

7.2. **The Gromov seminorm.** Gromov's work on bounded cohomology was to a large extend based on the fundamental observation that the cohomology of a cocomplex of Banach spaces with continuous codifferentials carries a canonical semi norm, and on exploiting such seminorms geometrically. More concretely, consider the standard homogeneous resolution

$$0 \to C_b^0(G)^G \to C_b^1(G)^G \to \dots$$

The space $C_b^n(G)$ are Banach spaces when equipped with the sup-norm, hence so are there closed(!) subspaces $C_b^n(G)^G$ and $\mathcal{Z}C_b^n(G)^G$. Unfortunately the subspace of coboundaries need not be closed in this space, whence $H_b^n(G;\mathbb{R})$ is in general not Hausdorff with respect to the quotient topology, let alone a Banach space. The best we can do is to consider the seminorm

$$\|\alpha\| := \inf\{\|c\|_\infty \,|\, c \in \alpha\}$$

on $H_b^n(G;\mathbb{R})$. Another drawback is that this semi norm is not canonical: We can easily write down injective resolution, which lead to a different semi norm on the level of cohomology. Indeed, it suffices to rescale the differentials. Despite all these shortcomings, the seminorm $\|\cdot\|$ is extremely useful. It is often referred to as the *Gromov seminorm*, although Gromov was by no means the first to consider it. (However, he was probably the first to provide substantial applications.)

One immediate property of the Gromov semi norm, which is extremely useful, is the following:

Proposition 7.2. *For every $\alpha \in H_{cb}^\bullet(G;\mathbb{R})$ and every $\rho : H \to G$ we have*

$$\|\rho^*\alpha\| \leq \|\alpha\|.$$

As we said, if we replace the homogeneous bar resolution by another injective resolution, then the induced isomorphism is in general not isometric. However, there are a couple of nice injective resolutions, which do indeed compute the same seminorm. We refer to these as *isometric resolutions*. For us the key example is as follows:

Proposition 7.3 (Burger-Monod, [9])**.** *The boundary resolution and the alternating boundary resolution are isometric.*

As an immediate consequence we deduce:

Corollary 7.4. *The Gromov seminorm on $H_b^2(G;\mathbb{R})$ is a norm for any locally-compact, second countable group G.*

The following section exploits some less obvious consequences of Proposition 7.3.

7.3. **The Gromov seminorm and transfer.** Throughout this section let $G = PU(1,1)$ and $\Gamma \cong h(\Gamma_g)$ be the image of Γ_g under a hyperbolization h, i.e. a cocompact lattice in G. We have seen above that the restriction map $H_{cb}^2(G;\mathbb{R}) \to H_b^2(\Gamma;\mathbb{R})$ has a left-inverse $T_b^2 : H_b^2(\Gamma;\mathbb{R}) \to H_{cb}^2(G;\mathbb{R})$ given by bounded transfer. From the explicit formula for T_b^2 we deduce:

Lemma 7.5. *The bounded transfer map $T_b^2 : H_b^2(\Gamma;\mathbb{R}) \to H_{cb}^2(G;\mathbb{R})$ is norm non-increasing, i.e. $\|T_b\alpha\| \leq \|\alpha\|$.*

We also recall that $H_{cb}^2(G;\mathbb{R}) = \mathbb{R}\cdot\kappa_b^G$, where κ_b^G is represented by $\frac{1}{2}\cdot o$ in the boundary resolution. We draw two consequences: Firstly, for every $\alpha \in H_b^2(\Gamma;\mathbb{R})$ there exists $\tau(\alpha) \in \mathbb{R}$ such that

$$T_b\alpha = \tau(\alpha)\cdot\kappa_b^G. \tag{7.1}$$

Secondly, we have $\|\kappa_b^G\| = \|\frac{1}{2}\cdot o\|_\infty = \frac{1}{2}$. We observe:

Proposition 7.6. *For every $\alpha \in H^2_b(G;\mathbb{R})$ we have*

$$|\tau(\alpha)| \leq 2\|\alpha\|.$$

Equality holds if and only if α is a multiple of $\kappa_b^{\Sigma_g}$.

Proof (cf. [5]*).* In view of Lemma 7.5 we have

$$\|\alpha\| \geq \|T_b\alpha\| = \left\| \frac{\tau(\alpha)}{2} \cdot o \right\| = \frac{1}{2} \cdot |\tau(\alpha)|,$$

which gives the desired inequality. Now assume that equality holds. We may assume without loss of generality that $\tau(\alpha) \geq 0$. Let c be a cocycle representing α in the boundary resolution. Then $\tau(\alpha) = 2\|c\|_\infty$, and thus

$$\|c\|_\infty \cdot o(x,y,z) = \tau(\alpha) \cdot \frac{o(x,y,z)}{2} = T_b^2 c(x,y,z) = \int_{\Gamma\backslash G} c(gx,gy,gz)dg,$$

which by G-invariance of β we can rewrite as

$$\int_{\Gamma\backslash G} \|c\|_\infty \cdot o(gx,gy,gz) - c(gx,gy,gz)dg = 0.$$

Assume now that the triple (x,y,z) is positively oriented. Then the above formula specializes to

$$\int_{\Gamma\backslash G} \|c\|_\infty - c(gx,gy,gz)dg = 0,$$

which implies $c(gx,gy,gz) = \|c\|_\infty$ for almost all g. Since g acts 3-transitively, this means that c is constant on positively oriented generic triples; combined with the antisymmetry it follows that c is almost everywhere a multiple of the orientation cocycle as claimed. □

We can actually express the number $\tau(\alpha)$ more explicitly:

Proposition 7.7. *For every $\alpha \in H^2_b(\Gamma;\mathbb{R})$ we have*

$$\tau(\alpha) = \frac{\langle c^2_\Gamma(\alpha), [\Sigma]\rangle}{|\chi(\Sigma_g)|}$$

Proof. There is a natural transfer map $T^2 : H^2(\Gamma;\mathbb{R}) \to H^2_c(G;\mathbb{R})$ in usual (continuous) cohomology, and the comparison map intertwines this with the bounded transfer discussed above. In particular, $T(c^2_\Gamma(\alpha)) = \tau(\alpha)\cdot\kappa^G = T(\tau(\alpha)\cdot\kappa^{\Sigma_g})$. Now since $\dim H^2(\Gamma;\mathbb{R}) = \dim H^2_c(G;\mathbb{R}) = 1$, the transfer map is an isomorphism, whence $c^2_\Gamma(\alpha) = \tau(\alpha)\cdot\kappa^{\Sigma_g}$, and thus

$$\langle c^2_\Gamma(\alpha), [\Sigma]\rangle = \tau(\alpha)\cdot\langle\kappa^{\Sigma_g}, [\Sigma]\rangle = \tau(\alpha)\cdot|\chi(\Sigma_g)|.$$

□

Combining both propositions we deduce:

Corollary 7.8. *Let $\alpha \in H^2_b(\Gamma;\mathbb{R})$. Then*

$$\langle c^2_\Gamma(\alpha), [\Sigma]\rangle \leq 2\cdot|\chi(\Sigma_g)|\cdot\|\alpha\|$$

with equality if and only if α is a multiple of κ_b^G.

Note that the proof of the inequality in Corollary 7.8 is essentially trivial: All the work went into the understanding of the equality case.

8. Maximal representations

8.1. Generalized Milnor-Wood inequality and maximal representations.

Throughout this section let G be a Hermitian simple Lie group and let $\rho : \Gamma_g \to G$ be a representation. The following result is due to Milnor [28][5] in the case $G = \mathrm{PU}(1,1)$ and due to Burger, Iozzi and Wienhard [8] in the general case.

Theorem 8.1 (Generalized Milnor-Wood inequality)**.** *The Toledo invariant satisfies*

$$T(\rho) \leq |\chi(\Sigma_g)| \cdot \mathrm{rk}(G).$$

Equality holds if and only if the following two conditions are satisfied:

(i) $\rho^*\kappa_b^G = \mathrm{rk}(G) \cdot \kappa_b^{\Sigma_g}$.
(ii) $\|\rho^*\kappa_b^G\| = \|\kappa_b^G\|$.

Proof. Combining Corollary 7.8 and Proposition 7.2 We have the chain of inequalities

$$\begin{aligned} T(\rho) &= \langle \rho^*\kappa_b^G, [\Sigma] \rangle \\ &\leq 2 \cdot |\chi(\Sigma_g)| \cdot \|\rho^*\kappa_b^G\| \\ &\leq 2 \cdot |\chi(\Sigma_g)| \cdot \|\kappa_b^G\| \\ &= 2 \cdot |\chi(\Sigma_g)| \cdot \|\kappa_b^G\|, \end{aligned}$$

Now the generalized Milnor-Wood inequality follows from the formula

$$\|\kappa_b^G\| = \frac{1}{2} \cdot \mathrm{rk}(G),$$

which is due to Clerc and Ørsted [15] in general and Domic and Toledo [16] for the classical groups. Equality in the generalized Milnor-Wood inequality holds if and only the two equalities $\langle \rho^*\kappa_b^G, [\Sigma] \rangle = 2 \cdot |\chi(\Sigma_g)| \cdot \|\rho^*\kappa_b^G\|$ and $\|\rho^*\kappa_b^G\| = \|\kappa_b^G\|$ hold. By Corollary 7.8 the former equality is equivalent to the existence of $\lambda \in \mathbb{R}$ wit $\rho^*\kappa_b^G = \lambda \cdot \kappa_b^{\Sigma_g}$, and then (ii) implies $\lambda = \mathrm{rk}(G)$. □

Motivated by this inequality we define:

Definition 8.2. A representation $\rho : \Gamma_g \to G$ is called a *representation of maximal Toledo invariant* (or *maximal representation* for short) if

$$T(\rho) = |\chi(\Sigma_g)| \cdot \mathrm{rk}(G).$$

It is called *weakly maximal* if there exists $\lambda \geq 0$ such that $\rho^*\kappa_b^G = \lambda \cdot \kappa_b^{\Sigma_g}$ and *tight* if $\|\rho^*\kappa_b^G\| = \|\kappa_b^G\|$.

Corollary 8.3. *A representation is maximal if and only if it is weakly maximal and tight.*

Combining this with Theorem 4.11 we deduce:

Theorem 8.4. *Every maximal representation (in fact, every weakly maximal representation of non-zero Toledo invariant) is discrete and faithful.*

In fact, maximal representations are Shilov-Anosov. We will not prove this here, though.

[5] There is a corresponding result in the non-orientable case due to Wood [30].

8.2. **Goldman's characterization of hyperbolizations.** Specializing to the case $G = \mathrm{PU}(1,1)$ we obtain:

Theorem 8.5. *Let $\rho : \Gamma_g \to \mathrm{PU}(1,1)$ be a representation. Then the following are equivalent:*

(i) *ρ is maximal.*
(ii) *$T(\rho) = |\chi(\Sigma)|$.*
(iii) *$\rho^*\kappa_b^G = \kappa_b^{\Sigma_g}$.*
(iv) *$\exists\lambda > 0$: $\rho^*\kappa_b^G = \lambda \cdot \kappa_b^{\Sigma_g}$.*
(v) *ρ is discrete, faithful and orientation-preserving.*
(vi) *ρ is a hyperbolization.*

Proof. The implications (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(v) have been discussed above in greater generality. The implication (v) $\Rightarrow$ (vi) is a classical result of Dehn, Nielsen and Baer, and (vi) $\Rightarrow$ (iii) holds by definition. $\square$

The equivalence (i)$\Leftrightarrow$(vi) was first established by Goldman in his thesis [21]. This was one of the starting points of higher Teichmller theory in general, and the theory of maximal representations in particular.

8.3. **Maximal representations and Higgs bundles.** One reason for the broad interest in the theory of maximal representations is the fact that they give rise to Higgs bundles, hence are amenable to harmonic map techniques. Given a representation $\rho : \Gamma_g \to G$ there is always a unique homotopy class of equivariant maps $\mathbb{D} \to \mathcal{X} = G/K$. However, this homotopy class of maps admits a harmonic representative if and only if the Zariski closure of $\rho(\Gamma_g)$ in G is a reductive subgroup. Therefore the following result of Burger, Iozzi and Wienhard is crucial for the theory of Higgs bundles:

Theorem 8.6 ([7])**.** *Maximal, and in fact all tight representations have reductive Zariski closure.*

The proof is another example for the use of cohomological techniques. In view of the cohomological definition of tight representations this is not surprising. In any case, this is a story for another time.

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INTRODUCTION TO POSITIVE REPRESENTATIONS (FOLLOWING FOCK-GONCHAROV)

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Abstract. In this talk, we will try to give a simple description of the set of positive representations of the fundamental group of a surface with non-empty boundary to the group $\mathrm{PSL}(m,\mathbb{R})$, as defined by Fock and Goncharov [1]. For each step of the construction, we will consider the classical cases $m = 2$ and $m = 3$ before turning to the general case. We will also give the main property of these representations, namely their faithfulness and discreteness.

1. Introduction

Given a surface $S_{g,s}$ of genus g with s boundary components, there are well-known coordinates on the (classical) Teichmüller space which are given by Thurston shearing coordinates. On the other hand, points in Teichmuller spaces can also be seen as conjugacy classes of representations of the fundamental group of the surface into the group $\mathrm{PSL}(2,\mathbb{R})$ satisfying various properties (discrete, faithful, etc ...). In this talk, we will show that it is possible to define complex coordinates on the bigger moduli space of representations into $\mathrm{PGL}(2,\mathbb{C})$ (in fact a finite ramified cover over this space) and see that Teichmuller space will be exactly the representations for which the coordinates are real positive. What is more interesting is that it is possible to generalize this picture to the case of $\mathrm{PGL}(m,\mathbb{C})$.

So the purpose of this talk will be to define the set of framed representations of a surface group into $\mathrm{PGL}(m,\mathbb{C})$ and define global coordinates on it using a triangulation of the surface. The coordinates will depend on the triangulation, but the set of representations with positive coordinates will be the same for all triangulations. This set of *positive representations* will be the so-called higher Teichmüller space in the sense of Fock and Goncharov. While the positive representations might be defined for a general split semi-simple real Lie group, using Lustzig notion of total positivity, we will restrict ourselves to the simpler case $G = \mathrm{PSL}(m,\mathbb{R})$ where everything can be made explicit.

To understand the properties of positive representations, we will see how to construct a representations from a given set of coordinates. We will also see that a positive representation defines a positive map from the Farey set of the surface to the flag variety, and that this notion can be generalized to closed surfaces

These notes constitute an introduction to some ideas of the definitions and properties of positive representations. We avoid technical difficulties in definitions and the proofs are only sketched here. For the correct definitions, and detailed proofs, one should obviously refer to the original articles of Fock and Goncharov [1, 2, 3].

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2. Framed representations

Let S be a compact orientable surface of genus g with $s \geq 1$ open discs removed. The standard presentation of the fundamental group is given by :

$$\pi_1(S) = \langle A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_s \mid \prod_{i=1}^{g}[A_i, B_i] \prod_{j=1}^{s} C_j \rangle$$

where the C_j correspond to the homotopy type of a curve going aroung the j-th hole. Let G be a Lie group. The moduli space of representations into G is

$$\mathcal{R}_G(S) = \mathrm{Hom}(\pi_1(S), G)/G$$

where the action of G is by conjugation. (Note that one should take the GIT quotient instead of the topological quotient to make this space Hausdorff, but we will avoid this discussion in order to simplify this exposition).

We are going to describe a ramified cover of the space $\mathcal{R}_G(S)$ which will be the space of *framed representations* and will be denoted $\mathfrak{X}_G(S)$. The additional data is given by choices of flags in $\mathbb{C}^m$, and hence we first recall the necessary vocabulary on flags in a vector spaces.

2.1. **Flags.**

Definition 2.1. *A (complete) flag F in a finite dimensional vector space V is an increasing sequence of subspaces of V such that*

$$\{0\} = F_0 \subset F_1 \subset \cdots \subset F_m = V$$

with $\dim F_k = k$.

A basis $(e_1, \ldots, e_m)$ is said to be adapted to the flag if the first k elements provide a basis of F_k. The stabilizer subgroup $\mathrm{Stab}(F) \subset G$ of a complete flag is identified with the set of invertible upper triangular matrices with respect to any basis adapted to the flag.

Let $F = (F_1, \ldots, F_m)$ and $F' = (F'_1, \ldots, F'_m)$ be two complete flags. We say that the pair of flags is in generic position if for any k we have $F_k \cap F'_{m-k} = \{0\}$. We say that a k-uple of flag is in generic position, if any pair of flag is in generic position.

Note that for a generic k-uple of flags $(A^{(1)}, \ldots, A^{(k)})$ in $\mathbb{C}^m$ and dimensions of subspaces $i_1, \ldots, i_k$, the dimension of a direct sum is given by

$$\dim(A^{(1)}_{i_1} + \cdots + A^{(k)}_{i_k}) = \min(m, i_1 + \ldots i_k)$$

Hence if $i_1 + \ldots i_k \leq m$ then the sum is direct.

The group $G = \mathrm{PSL}(m, \mathbb{R})$ or $\mathrm{PGL}(m, \mathbb{C})$ (depending on the field of the vector space V) acts naturally on flags by left multiplication. Obviously this action is transitive. The group G also acts on k-uples of flags. The space of G-orbits of k-uples of flags is called the space of configuration of k flags in $\mathbb{C}^m$, also denoted $\mathrm{Conf}_k(\mathbb{C}^m)$. If we restrict to k-uples of flags in generic position, the space is denoted $\mathrm{Conf}^*_k(\mathbb{C}^m)$.

Lemma 2.2. *The group G acts transitively on couple of flags in generic position. Moreover, the stabilizer of a couple of flag is the group of diagonal matrices in a basis adapted to one of the flag.*

Proof. Left as an exercise $\square$

2.2. **Definition.**

Definition 2.3. *A framed representation of S is given by the data of a representation $\rho \in \mathrm{Hom}(\pi_1(S), \mathrm{PGL}(m,\mathbb{C}))$ and flags $(F_1, \dots, F_s)$ in $\mathbb{C}^m$ associated to each connected component of ∂S, such that if $C_j \in \pi_1(S)$ corresponds to a boundary curve, then the corresponding flag is invariant by $\rho(C_j)$.*

The group $\mathrm{PGL}(m,\mathbb{C})$ acts naturally on framed representations by conjugation on the representation and left multiplication on the flags. Hence we define the moduli space of framed representation as the quotient of the set of framed representations by the action of G, and we denote it by $\mathfrak{X}_{\mathrm{PGL}(m,\mathbb{C})}(S)$.

A generic element $\rho(C_j) \in \mathrm{PGL}(m,\mathbb{C})$ corresponding to a boundary component have m invariant eigendirections. Hence there are $m!$ choices of invariant flags over a boundary circle corresponding to all possible ordering of the projective basis of eigendirections. However, when the eigenvalues are not all distinct (for example when the element is parabolic), then there are fewer choice of invariant flags, (and possibly only one). Hence, the space $\mathfrak{X}_{\mathrm{PGL}(m,\mathbb{C})}(S)$ is a ramified cover of $\mathbb{R}_{\mathrm{PGL}(m,\mathbb{C})}(S)$.

Remark 2.4. *One can extend this definition to surfaces with finite number of marked points removed on the boundary. In this case, the flags attached to the connected component of the boundary that are segments have no restrictions. This could allow us to identify a configuration of k flags as a framed representation of a disc with k marked points on the boundary.*

2.3. **Triangulations of surfaces.** Let S be a surface of genus g with s boundary components, such that $\chi(S) = 2-2g-s < 0$. Shrink all boundary components to points. Then the surface S can be cut into triangles with vertices at the shrunk boundary components. We call it an *ideal triangulation* of S.

Let T be a triangulation, and $V(T), E(T), F(T)$ the set of vertices, arcs, external arcs and faces of the triangulation T. Simple computation using the Euler characteristic give

(1) $|V(T)| = s$
(2) $|E(T)| = 6g-6+3s$
(3) $|F(T)| = 4g-4+2s$

3. Coordinates on moduli space of framed structure

We are now going to describe coordinates for the spaces $\mathfrak{X}_G(S)$. One of the main idea to find coordinates is to use a triangulation and decompose the space into a product of simpler space, that will correspond to moduli spaces of triples of flags and quadruples of flags. Hence, the coordinates are divided into two groups :

- The triangle invariants which corresponds to coordinates on $\mathrm{Conf}_3^*(\mathbb{C}^m)$.
- The edge invariants which parametrize the possible gluing of two triple of flags along an edge. We will show that this is parametrized by a maximal torus in $\mathrm{PGL}(m,\mathbb{C})$.

Theorem 3.1. *Let S be a surface and T an ideal triangulation. There is a birational isomorphism :*

$$\pi_T : \mathfrak{X}_G(S) \longrightarrow \prod_{t \in F(T)} (\mathrm{Conf}_3^*(\mathbb{C}^m)) \times \prod_{e \in E(T)} H$$

where H is a maximal torus in $\mathrm{PGL}(m,\mathbb{C})$.

The demonstration of this theorem will occupy this section where the map π_T will be explicitly constructed, and the next section where π_T^{-1} will be determined.

We will first study the classical case of $\mathrm{PGL}(2,\mathbb{C})$ where the invariants will be related to Thurston shearing coordinates on Teichmüller space. Then, we will see the case of $\mathrm{PSL}(3,\mathbb{C})$ where one has to understand the triple-ratio of a triple of flags in $\mathbb{C}^3$. The general coordinates will easily be constructed from this two basic cases.

3.1. **Cross-ratios in the $m = 2$ case.** A flag in $\mathbb{C}^2$ is simply given by a direction in $\mathbb{C}^2$ or equivalently a point in $\mathbb{C}P^1$. A classical result states that $\mathrm{PGL}(2,\mathbb{C})$ acts transitively on generic triples of flags in $\mathbb{C}P^1$, so there is only one orbit of generic triples of flags. Hence, the space $\mathrm{Conf}_3^*(\mathbb{C}^2)$ is reduced to a single point. Note that a maximal torus in $\mathrm{PGL}(2,\mathbb{C})$ is isomorphic to $\mathbb{C}^*$. Hence we simply have to find a unique coordinate for the configuration of four flags in $\mathbb{C}P^1$.

Let A, B, C, D be four flags in $\mathbb{C}^2$ in generic position, assimilated to points in $\mathbb{C}P^1$. We can consider that these four flags are attached to the vertices of a quadrilateral. Take the triangulation of this quadrilateral with an edge joining (the vertices associated to) A and C. We are going to associate a coordinate to this configuration and attach it to the edge AC.

The coordinate is given by he cross ratio of the 4 points (A, B, C, D) (in the correct order) in $\mathbb{C}P^1$. denoted by

$$[A, B, C, D] = \frac{(A - D)(B - C)}{(A - C)(B - D)}$$

This definition is equivalent to the following one : send the first triangle (A, B, C) to $0, -1, \infty$ (which is possible as the action is transitive) and then the fourth vertex is sent to a point $x \in \mathbb{C}P^1$ which is exactly this cross-ratio. Note that this definition is not exactly the standard definition, but there is a sign-change. This is due to the fact that we want the cross-ratio to be positive in specific situations.

Now let S be a general surface. Let $[\hat{\rho}]$ be a conjugacy class of framed representation and T a triangulation of S. We can lift this triangulation to a triangulation $\widetilde{T}$ of the universal cover $\widetilde{S}$. Starting with a choice of an arbitrary choice of a flag at a vertex of $\widetilde{T}$ we can associate to each vertex of the triangulation $\widetilde{T}$ a flag in a ρ-equivariant way. We can associate a coordinate for each edge of this triangulation by computing the cross-ratio of the four-points defined by the edge. We can restrict ourselves to a fundamental domain of the universal cover, as the ρ-equivariance ensures that the coordinates are invariant by deck transformation. Hence we have the map

$$\Phi_T : \mathfrak{X}_{\mathrm{PGL}(2,\mathbb{C})}(S) \longrightarrow (\mathbb{C}^*)^{|E(T)|}$$

that will give a set of coordinates on the moduli space of framed representations.

3.2. **Triple ratio in the $m = 3$ case.** We need to understand the moduli space of configuration of triples of flags in $\mathbb{C}^3$, and find coordinates on it. A flag in $\mathbb{C}^3$ is given by a line and a plane, or equivalently by a point and a line in $\mathbb{C}P^2$. Heuristically, the space of triples of flags is of dimension 9 (dimension 3 for each flag) while the group $G = \mathrm{PGL}(m,\mathbb{C})$ is of dimension 8. Hence the moduli space of configuration is of dimension 1, and we shall need only one invariant to describe it. This invariant is given by the triple ratio defined as follows.

Let $A = (A_1, A_2)$, $B = (B_1, B_2)$ and $C = (C_1, C_2)$ be a triple of flag, where $\dim A_i = i$. Choose v_A, v_B, v_C some direction vectors for the lines A_1, B_1 and C_1. and let $f_a, f_b, f_c \in (\mathbb{C}^3)^*$

be linear forms representatives defining the planes A_2, B_2 and C_2. The triple ratio is given by :

$$X := r_3(A, B, C) = \frac{f_a(v_b)f_b(v_c)f_c(v_a)}{f_a(v_c)f_b(v_a)f_c(v_b)}$$

This does not depend on the choices of the vectors or the linear form (easy exercise). This quantity is a complete invariant of generic configuration of three flags. Namely, if two flags A, B, C and A', B', C' have the same triple ratio, then there exists a unique element $g \in \mathrm{PGL}(m, \mathbb{C})$ such that $g(A) = A'$, $g(B) = B'$ and $g(C) = C'$ (left as an exercise).

Now we can look at the moduli space of configuration of four flags in $\mathbb{C}^3$. Let A, B, C, D be four flags in $\mathbb{C}^3$ in generic position and such that we have $r_3(A, B, C) = X$ and $r_3(A, C, D) = Y$. We want to construct invariants using cross-ratio of quadruples of points in $\mathbb{C}P^1$ associated to the edge AC. Note that four planes in $\mathbb{C}^3$ sharing a line in common define four lines in $\mathbb{C}^2$ by projection, and hence can be turned into four points in $\mathbb{C}P^1$.

The four planes $(A_2, A_1 \oplus B_1, A_1 \oplus C_1, A_1 \oplus D_1)$ all contain the line A. Hence, we can define their cross-ratio Z and get an element in $\mathbb{C}^*$. Similarly the four planes $(C_2, C_1 \oplus D_1, C_1 \oplus A_1, C_1 \oplus B_1)$ all contain the line C_1. So we can also define their cross-ratio W.

Proposition 3.2. *The map*

$$\begin{aligned} \mathrm{Conf}_4^*(\mathbb{C}^3) &\longrightarrow (\mathbb{C}^*)^4 \\ (A, B, C, D) &\longmapsto (X, Y, Z, W) \end{aligned}$$

is a birational isomorphism.

Now let S be a general surface. Let $[\hat{\rho}]$ be a conjugacy class of framed representation and T an ideal triangulation of S. Again on each vertex of the lifted triangulation, we can associate the flag of the corresponding element of ∂S in a ρ-equivariant way. Hence, for each triangle of the triangulation we can associate a coordinate by computing the triple-ratio of the triple of flags associated to the vertices. And for each edge of the triangulation, we can associate the two coordinates corresponding to the two cross-ratio as above. Hence we have a map

$$\Phi_T : \mathfrak{X}_{\mathrm{PGL}(3,\mathbb{C})}(\hat{S}) \longrightarrow (\mathbb{C}^*)^{2|E(T)|+|F(T)|}$$

which gives $16g - 16 + 8s$ coordinates for the moduli space of framed representations.

3.3. **Coordinates in the general case.** When $m \geq 3$ the same ideas are used to construct coordinates. The triangle invariants are computed by restricting the flags in $\mathbb{C}^m$ to $\mathbb{C}^3$ and the edge invariants are computed by finding four planes sharing a line in $\mathbb{C}^3$.

3.3.1. *Invariants of triples of flags.* Let $m \geq 3$ and A, B and C be a generic triple of flags in $\mathbb{C}^m$. Denote by A_k the subspace of dimension k in the flag A. The main idea to get invariants of triple of flags is to extract from the three flags in $\mathbb{C}^m$, a certain number of triples of flag in $\mathbb{C}^3$ in a clever way.

Let $i, j, k \geq 1$ such that $i + j + k = m$. The direct sum $W_{i,j,k} = A_{i-1} \oplus B_{j-1} \oplus C_{k-1}$ is a subspace of dimension $m - 3$. Hence the quotient $V_{i,j,k} = \mathbb{C}^m / W$ is isomorphic to $\mathbb{C}^3$.

We can take the projection of the subspace A_i and A_{i+1} on V, and this give a flag $A^{(i,j,k)} = (\overline{A}_i, \overline{A}_{i+1})$ in $V_{i,j,k}$. Similarly, we get two other flags $B^{(i,j,k)}$ and $C^{(i,j,k)}$ by projecting B and C on $V_{i,j,k}$. Hence we have a triple of flags in $\mathbb{C}^3$ and hence we can compute :

$$X_{i,j,k}(A,B,C) = r_3(A^{(i,j,k)}, B^{(i,j,k)}, C^{(i,j,k)})$$

Obviously, this quantity is an invariant of triples of flags in $\mathbb{C}^m$. For each $i+j+k=m$ we get another invariant. This gives $\frac{(m-1)(m-2)}{2}$ coordinates in $\mathbb{C}^*$. We see that this number should be equal to the dimension of the space of configuration of three flags in $\mathbb{C}^m$. Indeed we have

Proposition 3.3. *The map*

$$\begin{aligned} \mathrm{Conf}_3(\mathbb{C}^m) &\longrightarrow (\mathbb{C}^*)^{\frac{(m-1)(m-2)}{2}} \\ (A,B,C) &\longmapsto \{X_{i,j,k}(A,B,C) | i,j,k \geq m, \ \ \text{and } i+j+k=m, \} \end{aligned}$$

is a birational isomorphism.

3.3.2. *Invariants of quadruples of flags.* Let A,B,C,D be a quadruple of flags in $\mathbb{C}^m$. The main idea is to extract several quadruples of lines in $\mathbb{C}^2$, that we will associate to this edge

Let $i,j \geq 1$ such that $i+j=m$. The direct sum $U_{i,j} = A_{i-1} \oplus C_{j-1}$ is a subspace of dimension $m-2$. Hence the quotient $V_{i,j} = \mathbb{C}^m / U_{i,j}$ is isomorphic to $\mathbb{C}^2$

We can take the projection of A_i, B_1, C_j and D_1 that we denote $A^{(i,j)}$, $B^{(i,j)}$, $C^{(i,j)}$ and $D^{(i,j)}$ on this subspace. And this will give a quadruple of lines in $\mathbb{C}^2$ so we can compute :

$$X_{i,j}(A,B,C,D) = [A^{(i,j)}, B^{(i,j)}, C^{(i,j)}, D^{(i,j)}]$$

This quantity is an invariant of quadruple of flags in $\mathbb{C}^m$. For each $i+j=m$ we get another invariant. This gives $m-1$ coordinates in $\mathbb{C}^*$ that we attach to an edge. This is exactly the dimension of a maximal torus H in $\mathrm{PGL}(m,\mathbb{C})$.

3.3.3. *Coordinates for a general surface.* Let S be a general surface, with a triangulation. To visualize easily all the invariants and their relation, we use an m-triangulation of the surface S, where each original triangle is divided into more triangles as seen on the figure below. Each vertex of this m-triangulation correspond to a triple (i,j,k) such that $i,j,k \geq 0$ are integers and $i+j+k=m$.

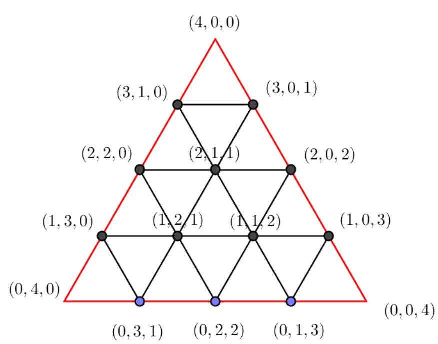

FIGURE 1. 4-triangulation : Each point corresponds to a triple (i,j,k) where at least two are non-zero

For each point, except the vertices of the original triangle, on this m-triangulation, we associate a coordinate. Namely, to each vertex in the interior of a triangle corresponds a triple $i, j, k \geq 1$ such that $i + j + k = m$ so we can associate a triangle invariant. and for each vertex in the interior of a side of the triangle corresponds a couple $i, j \geq 1$ such that $i + j = m$ so we can associate an edge invariant.

Let $N(S, m)$ the number of vertices of this m-triangulation of the surface. We get

$$N(S, m) = |F(T)| \left(\frac{(m-1)(m-2)}{2} \right) + |E(T)|(m-1)$$

So with the same principle as before, this gives a map

$$\Phi_T : \mathfrak{X}_{\mathrm{PGL}(m,\mathbb{C})}(S) \longrightarrow (\mathbb{C}^*)^{N(S,m)}$$

In the next section, we construct the inverse map Φ_T^{-1}. We start with coordinates in $(\mathbb{C}^*)^{N(S,m)}$ and we find an element of $\mathfrak{X}_{\mathrm{PGL}(m,\mathbb{C})}(S)$.

4. Construction of a framed representation from coordinates

4.1. **General Strategy.** Starting from the triangulation, we construct a graph Γ embedded into the surface by drawing small edges transversal to each side of the triangles and inside each triangle connect the ends of edges pairwise by three more edges. Orient the small edges in the triangles in counterclockwise direction and the long edges crossing the triangulation in an arbitrary way (see Figure 2).

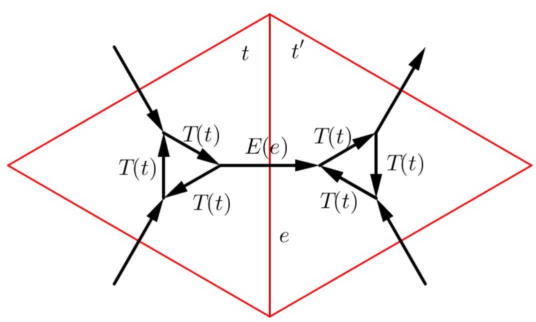

Figure 2. The oriented graph dual to the triangulation

To construct a framed representation, we assign to every oriented edge $e \in \Gamma$, an element in $\mathrm{PGL}(m, \mathbb{C})$ that will depend on the coordinates associated to it.

For any long edge $e \in \Gamma$, we associate a matrix $E(\{X_{i,j}\})$ which will depend on the edge coordinates $X_{i,j}$ with $i + j = m$ of the corresponding edge of the triangulation. And to any small edge $e \in \Gamma$, and we associate the matrix $T(\{X_{i,j,k}\})$ which will depend on triangle coordinates of the triangle in which it stands.

Given this, for any closed path $\gamma \in \pi_1(S)$ on the surface, there is a path on Γ homotopic to it. So one can assign to γ an element of $\mathrm{PGL}(m, \mathbb{C})$ by multiplying the group elements (or their inverse if the path goes along the edge against its orientation) assigned to the edges followed by the path. When the matrices T and E are well constructed, this defines a framed representation of the fundamental group of S, with the appropriate coordinates.

The main idea is the following. Let t be a triangle with flags (A, B, C) at each of its vertices and with invariants $X_{i,j,k}$. Then the matrix $T(t)$ associated to this triangle will correspond to the element of $\mathrm{PGL}(m, \mathbb{C})$ that will perform the cyclic permutation of the flags $(A, B, C) \mapsto (B, C, A)$.

Similarly, if (A, B, C, D) is a quadruple of flags with coordinates $X_{i,j}$ associated to the edge (A, C). Then the matrix $E(e)$ will correspond to the element of $\mathrm{PGL}(m, \mathbb{C})$ that sends (A, C) to (C, A) and moreover sends B_1 to D_1.

We will see explicit constructions in the cases $m = 2$ and $m = 3$. The explicit construction in the general case is a little more difficult to compute but follow the same principle.

4.2. **The case** $m = 2$**.** Let (A, B, C) be a triple of flags in $\mathbb{C}^2$ or equivalently, three points in $\mathbb{C}P^1$. For the matrix T, we take the element sending the triple (A, B, C) to (B, C, A), and we can choose (A, B, C) to be $(0, \infty, -1)$ so that

$$T = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$

Now let (A, B, C, D) be a quadruple of points such that the coordinates associated to the edge AC is equal to $z \in \mathbb{C}^*$. For convenience, we may take $(A, B, C, D) = (0, -1, \infty, z)$. Now the matrix $E(z)$ should be the element sending $(0, \infty)$ to $(\infty, 0)$ and moreover sending -1 to z. This is

$$E(z) = \begin{pmatrix} 0 & z \\ -1 & 0 \end{pmatrix}$$

This will gives the representation of $\pi_1(S)$.

To recover the framing over holes, notice that for any coordinate z a product of the form $E(z)T$ is a lower-triangular matrix (while $E(z)T^{-1}$ is upper lower triangular. When following a path around the boundary we get only one type of product, because we always turn right (resp. left), and hence only get T (resp. T^{-1}) in the formulas. So the monodromy $\rho(C_j)$ is upper-triangular (or lower-triangular), which gives a preferred eigendirection for each boundary, and hence an invariant flag.

4.3. **The case m=3.** Let A, B, C be a triple of flag in $\mathbb{C}^3$, with triple ratio $r_3(A, B, C) = X$. We want to construct an element g that will send (A, B, C) on (B, C, A).

By the transitivity of the G-action on couple of flags, we can assume without loss of generality that we have a basis (e_1, e_2, e_3) such that :

$$A_1 = e_1, \qquad A_2 = e_1 \oplus e_2, \qquad C_1 = e_3, \qquad C_2 = e_3 \oplus e_2$$

and moreover such that

$$B_1 = e_1 - e_2 + e_3$$

In this case, a linear form generating B_2 is given by $Xe_1^* + (1 + X)e_2^* + e_3^*$. So the desired matrix in the basis (e_1, e_2, e_3) is given by :

$$T(X) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ X & 1 + X & 1 \end{pmatrix}$$

Now let A, B, C, D be four flags in $\mathbb{C}^3$ such that the invariant associated to the edge AC are Z and W. Again, we can assume that we have a basis such that A and C are the standard flags $(e_1, e_1 \oplus e_2)$ and $(e_3, e_3 \oplus e_2)$ respectively. And with B_1 generated by $e_1 - e_2 + e_3$. Then by definition of the two cross-ratios Z and W we have that the vector $Z^{-1}e_1 + e_2 + We_3$ is a direction vector of D_1. Then in this basis, the element sending (A, C) to (C, A) and B_1 to D_1 is given by

$$E(Z, W) = \begin{pmatrix} 0 & 0 & Z^{-1} \\ 0 & -1 & 0 \\ W & 0 & 0 \end{pmatrix}$$

Remark 4.1. *The general case is obtained by the same principle. However the matrices and their computations are a little too long to be included in these notes. The interested reader can find all the details in* [1] **9.8**.

5. Positive Representations

We now have all the tools to define the set of positive representations as a subset of the space of framed representations.

5.1. **Positive part of the moduli space.** The coordinates constructed depend on the chosen triangulation. An important thing is to understand how these coordinates change when we choose a different triangulation of the surface S. A classical result states that two triangulation are related by a sequence of flips. Hence we need to understand what happens to the coordinates when we perform a flip along an edge of a triangulation.

In the case $m = 2$. It is an easy exercise to recover the following formulas in Figure 3.

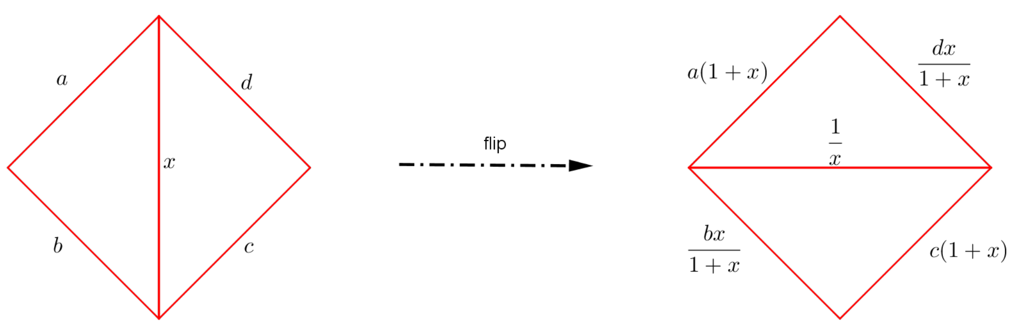

Figure 3. Change of coordinates after a flip

5.1.1. *Cluster mutations.* There is a simple interpretation of these coordinate changes in terms of cluster algebra. From a triangulation of the surface S, we can define an oriented graph (a quiver) associated to it (see Fig. 4). Flipping along an arc of the triangulation is equivalent to a performing a mutation of the quiver along a vertex. Such mutation are defined for any quiver as follows :

Let Q be a quiver and k one of its vertex, the mutated quiver $\mu_k(Q)$ is obtained by performing the three following steps :

(1) For each path of length two of the form $i \longrightarrow k \longrightarrow j$, add an arrow $i \longrightarrow j$
(2) Reverse each arrow incident to the vertex k.

(3) Erase any 2-cycles that could have been created by previous steps.

The new coordinates are given by the so-called mutations formulas, which were discovered by Fomin and Zelevinsky in a much broader context, and depend only on the quiver. If $(x_1, \ldots, x_n)$ are the coordinates associated to each vertex of the quiver Q. Let $\varepsilon(i,j)$ be the number of oriented arrows between the vertex i and j. Then the new coordinates $(x'_1, \ldots, x'_n)$ of the quiver $\mu_k(Q)$ are given by

$$x'_j = \begin{cases} \dfrac{1}{x_j} & \text{if } j = k, \\ x_j(1+x_k)^{\varepsilon(j,k)} & \text{if } \varepsilon(j,k) \geq 0, \\ x_j(1+x_k^{-1})^{\varepsilon(j,k)} & \text{if } \varepsilon(j,k) < 0 \end{cases}$$

For a detailed exposition of the relation between cluster algebras and triangulations of hyperbolic surfaces, one should refer to the article [4] of Fomin, Shapiro and Thurston.

5.1.2. *Flips as sequence of mutations.* In the general case, the explicit formulas for the change of coordinates after a flip could be computed directly but would be quite heavy. However, it is again possible to interpret these changes in terms of cluster mutation. We can associate a quiver to any m-triangulation of the surface as indicated in the figure 4. In this case, changing the triangulation is equivalent to a particular sequence of quiver mutations, and the change of coordinates corresponds to the associated cluster mutations (see Chapter **10** of [1]).

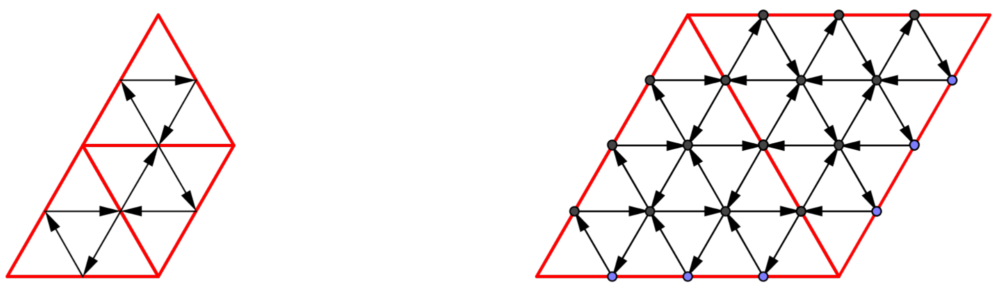

FIGURE 4. Quiver associated to a triangulation and to an m-triangulation of the surface

We notice that the formulas for cluster mutations are subtraction-free. Hence, any change of triangulation gives a subtraction-free change of coordinates. This defines a *positive atlas* on the space of framed structure (the actual definition of a positive atlas is not needed). Hence, it is clear that if a framed representation has positive coordinates in a given triangulation, then it will have positive coordinates with respect to any triangulation. This allows us to define

Definition 5.1. *A framed representation is said to be* positive *if its coordinates with respect to one (and hence any) triangulation are all real positive.*

The set of positive framed representation is denoted $\mathfrak{X}^+_G(S)$.

A positive representation is just the image of a positive framed representation through the forgetful map $\mathfrak{X}_G(S) \to \mathcal{R}_G(S)$.

In the next two paragraphs, we will give sketches of proofs of the fundamental properties of positive representations. Namely that they are faithful and discrete.

5.2. **Totally positive matrices and faithfulness of positive representations.**

Proposition 5.2. *Let $[\rho]$ be a positive framed representation. Then the corresponding representation ρ is faithful. Moreover, for any element $\gamma \in \pi_1(S)$ which is non-trivial and non-boundary, the element $\rho(\gamma)$ is positive hyperbolic.*

To prove this proposition, we must look at the construction of the framed structure from the coordinates. Let γ be a non-trivial non-boundary loop. Then the element $\rho(\gamma)$ is of the form

$$\rho(\gamma) = E(e_1)(T(t_1))^{\epsilon_1} E(e_2)(T(t_2))^{\epsilon_2} \cdots E(e_p)(T(t_p))^{\epsilon_p}$$

Suppose all the coordinates are positive. Then we notice that any matrix of the form $E(e)T(t)$ (resp. $E(e)(T(t))^{-1}$) will be a totally positive upper-triangular matrix (resp. totally positive lower triangular matrix). The product of totally positive matrices is a totally positive matrix. And totally positive matrices have the property to have real, positive and distinct eigenvalues. Hence, $\rho(\gamma)$ is positive hyperbolic. The faithfulness is easily deduced from that.

5.3. **Farey set and discreteness.** To prove discreteness, we need to introduce a new object.

Definition 5.3. *The Farey set of the surface S, denoted $\mathcal{F}_\infty(S)$ is a cyclic $\pi_1(S)$-set defined as follows.*

Shrink the holes of S to punctures and lift to the universal cover, which is an open disc. The set $\mathcal{F}_\infty(S)$ is the set of lifts of punctures which is a countable subset on the boundary of the disc. The group $\pi_1(S)$ acts naturally by deck transformation, and the cyclic structure is given by the cyclic structure on the boundary of the disc.

We can defined framed representations using only maps from the Farey set to the flag variety as follows as there is a natural bijection between $\mathfrak{X}_{\mathrm{PGL}(m,\mathbb{C})}(S)$ and the set of ρ-equivariant maps

$$\beta_\rho : \mathcal{F}_\infty(S) \longrightarrow \mathrm{Flag}(\mathbb{C}^m)$$

modulo conjugation by $\mathrm{PGL}(m, \mathbb{C})$.

Definition 5.4. *A triplet of flags (A, B, C) is* positive *if it is equivalent (modulo the action of* $\mathrm{PGL}(m, \mathbb{C})$*) to a triple $(F^+, F^-, u \cdot F^-)$ where F^+ and F^- are the standard flag and u is a totally positive upper triangular matrix.*

A configuration of flag $(F_1, \ldots, F_n)$ is positive if any oriented triplet of flag is positive.

If $[\rho]$ is a positive framed representation, then the map β_ρ takes values in $\mathrm{Flag}(\mathbb{R}^m)$ and is a positive map in the following sense : for any finite cyclic subset $(x_1, \ldots, x_k) \in \mathcal{F}_\infty(S)$, the configuration of flags $(\beta(x_1), \ldots, \beta(x_n))$ is positive.

Proposition 5.5. *We have an identification of $\mathfrak{X}^+_{\mathrm{PGL}(m,\mathbb{C})}(S)$ and the set of (ρ, β) where ρ is a representation and $\beta : \mathcal{F}_\infty(S) \to \mathrm{Flag}(\mathbb{R}^m)$ is a positive $(\pi_1(s), \rho)$-equivariant map, modulo conjugation.*

This characterization of positive representation can be extended to surfaces without boundaries. In this case, there are no positive structure or coordinates but the properties of positive representations remain.

The identification also allows us to prove the following theorem

Theorem 5.6. *Positive representations are discrete.*

Proof. Let ρ be a positive representation. Let $(a,b,c) \in \mathcal{F}_\infty(S)$ be an ideal triangle of the triangulation of S. The flag $\beta(b)$ belongs to the set

$$\mathcal{D}_- = \{F \in \mathrm{Flag}(\mathbb{R}^m) | (\beta(a), F, \beta(c)) \text{ is a positive configuration of flags}$$

This set is open and disjoint from

$$\mathcal{D}_+ = \{F \in \mathrm{Flag}(\mathbb{R}^m) | (\beta(a), \beta(c), F) \text{ is a positive configuration of flags}$$

Hence, there exist an open neighborhood $O \subset \mathrm{PSL}(m, \mathbb{R})$ such that for all $g \in O$, we have $g \cdot \beta(b) \notin \mathcal{D}_+$.

Now let $\gamma \in \pi_1(S)$ and suppose that the image $b' = \gamma \cdot b$ is contained in the segment $(a,c) \in \mathcal{S}^1$. As β_ρ is a positive map, the quadruple of flags $(\beta(a), \beta(b), \beta(c), \rho(\gamma)\beta(b))$ is positive. We deduce easily that $\rho(\gamma) \notin O$. $\square$

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Abstract

These are some brief notes on surface groups actions on complex hyperbolic space, mainly work of Domingo Toledo, and generalizations.

1 Hyperbolic space

We begin with motivation from hyperbolic space. First, the definition most useful to our generalization. Let q be the quadratic form on $\mathbb{R}^{n+1}$ given by

$$q(\alpha_1, \dots, \alpha_{n+1}) = \alpha_1^2 + \cdots + \alpha_n^2 - \alpha_{n+1}^2.$$

The associated symmetric matrix is

$$B = \mathrm{diag}(1, \dots, 1, -1),$$

and we also use B to denote the associated symmetric bilinear form on $\mathbb{R}^{n+1}$. The *Klein model* of hyperbolic space $\mathbf{H}^n$ is then

$$\mathcal{P}^n = \left\{ \text{lines } \ell \subset \mathbb{R}^{n+1} \ : \ q|_\ell \text{ is negative} \right\} \subset \mathbb{RP}^n.$$

The metric[1] is

$$d(x, y) = 2\cosh^{-1}\left(\frac{B(u, v)}{\sqrt{q(u)q(v)}} \right),$$

where $u, v \in \mathbb{R}^{n+1}$ are vectors representing the lines $x, y \in \mathbb{RP}^n$. Notice that $q(u), q(v) < 0$, but the square root is real.

[1] The factor of 2 makes the curvature -1.

Exercise 1. Show that d is well-defined.

Exercise 2. Identify $\mathcal{P}^n$ with the unit ball and upper half space models.

Exercise 3. Show that the ideal boundary of $\mathcal{P}^n$ is the set of *q-isotropic* lines, $\{\ell \subset \mathbb{R}^{n+1} \ : \ q|_\ell = 0\}$.

Exercise 4. Show that the (orientation-preserving) isometry group of $\mathcal{P}^n$ is exactly $\mathrm{PO}_0(n,1)$, the connected component of 1 in the group of projective transformations of $\mathbb{R}^{n+1}$ that fix q.

Exercise 5. When is $\mathrm{PO}(n,1)$ connected?

Now, we consider a (closed) surface group Σ (of genus $g \geq 2$) and homomorphisms $\Sigma \to \mathrm{PO}_0(n,1)$. We start with the case $n = 2$. Fix a hyperbolic structure, i.e., a discrete and faithful representation $\rho_0 : \Sigma \to \mathrm{PO}_0(2,1)$, and let $S = \mathbf{H}^2 / \rho_0(\Sigma)$. The following exercise (see [6]) is important motivations for our generalizations to complex hyperbolic space.

Exercise 6. Given $\rho : \Sigma \to \mathrm{PO}_0(2,1)$, build a ρ-equivariant $\mathbf{H}^2$-bundle E_ρ over S with fiber $\mathbf{H}^2$ with the following property. Let vol_ρ be the $\mathrm{PO}_0(2,1)$-invariant extension of the volume form on $\mathbf{H}^2$ to E_ρ by pushing it along fibers. Then given any section σ of E_ρ the number

$$\mathrm{vol}(\rho) = \int_S \sigma^* \mathrm{vol}_\rho$$

is independent of σ.

In this thesis [5], Goldman showed that $\mathrm{vol}(\rho)$ is related to the Euler number of a certain bundle. Using this, he proved the following.

Theorem 1. *Let Σ be the fundamental group of a closed surface S of genus $g \geq 2$, and $\rho : \Sigma \to \mathrm{PO}_0(2,1)$ any representation. Then*

1. $\mathrm{vol}(\rho) \in 2\pi\mathbb{Z}$;
2. $|\mathrm{vol}(\rho)| \leq 2\pi(g-1)$;
3. $|\mathrm{vol}(\rho)| = 2\pi(g-1) = \mathrm{vol}(S)$ *if and only if ρ is Fuchsian, i.e., ρ is the holonomy of a complete hyperbolic structure on S;*
4. $\mathrm{vol}(\rho_1) = \mathrm{vol}(\rho_2)$ *if and only if ρ_1 can be deformed into ρ_2, that is, if and only if ρ_1 and ρ_2 lie in the same connected component of* $\mathrm{Hom}(\Sigma, \mathrm{PO}_0(2,1))$.

This is a remarkable rigidity theorem, especially for something as non-rigid as the fundamental group of a surface. The representations with $|\mathrm{vol}(\rho)| < 2\pi(g-1)$ are very poorly understood. That the extremal representations are the most rigid and geometrically significant[2] is a recurring theme in these notes.

Before complexifying, we consider $\mathbf{H}^n$ for higher n. To start, there are some easy representations $\Sigma \to \mathrm{PO}_0(n,1)$, namely, representations that factor

$$\Sigma \to \mathrm{SO}_0(2,1) \to \mathrm{PO}_0(n,1)$$

given by a totally geodesic embedding of $\mathbf{H}^2$ into $\mathbf{H}^n$.

Exercise 7. Explain why $\mathbf{H}^2 \to \mathbf{H}^n$ gives an embedding $\mathrm{SO}(2,1) \to \mathrm{PO}(n,1)$ instead of $\mathrm{PO}(2,1) \to \mathrm{PO}(n,1)$. Hint: Think about how the scalar matrices in $\mathrm{O}(n,1)$ intersect, say, $\mathrm{SO}(2,1)$. Explain this in geometric terms via the normal bundle to $\mathbf{H}^2$ in $\mathbf{H}^n$.

Exercise 8. When does a representation $\rho : \Sigma \to \mathrm{PO}_0(2,1)$ lift to a representation $\Sigma \to \mathrm{SO}(2,1)$?

If the map $\Sigma \to \mathrm{SO}_0(2,1)$ is (the lift of) a Fuchsian representation of Σ, then we also call the representation $\Sigma \to \mathrm{PO}_0(n,1)$ Fuchsian. That is, Fuchsian representations of Σ into $\mathrm{PO}_0(n,1)$ are extensions of Fuchsian representations into $\mathrm{SO}_0(2,1)$ via a totally geodesic embedding $\mathbf{H}^2 \to \mathbf{H}^3$.

Exercise 9. Describe the totally geodesic embeddings of $\mathbf{H}^m$ into $\mathbf{H}^n$ $(m < n)$ via subspaces of $\mathbb{R}^{n+1}$ on which the restriction of q is positive definite.

Fuchsian representations of Σ into $\mathrm{PO}_0(n,1)$ are, however, not rigid for $n \geq 3$. This is due to the existence of *bending deformations*. Here is the sketch:

1. Write Σ as an amalgamated product $G_1 *_{\langle\gamma\rangle} G_2$ over $\mathbb{Z}$ by splitting the surface S along an essential separating curve γ.
2. Take a Fuchsian representation $\rho : \Sigma \to \mathrm{SO}_0(2,1) \to \mathrm{PO}_0(n,1)$ with associated totally geodesic embedding $f : \mathbf{H}^2 \to \mathbf{H}^n$. Let $\widetilde{\gamma} \subset \mathbf{H}^2$ be the axis of γ.
3. Let $\mathcal{F}$ be a fundamental domain for the action of Σ on $\mathbf{H}^2$ such that $\widetilde{\gamma}$ cuts $\mathcal{F}$ into two subdomains $\mathcal{F}_1$ and $\mathcal{F}_2$ associated with each side of γ in S.
4. Let $f_t : \mathbf{H}^2 \to \mathbf{H}^n$ be a continuous family of totally geodesic embeddings $(t \in \mathbb{R})$ with $f_0 = f$ and $f_t(\widetilde{\gamma}) = f_0(\widetilde{\gamma})$.

[2] I think it's fair to say that the Fuchsian representations into $\mathrm{PO}_0(2,1)$ are the most significant representations of Σ.

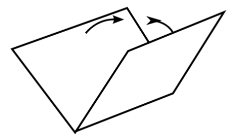

Figure 1: Local picture of a bending with $\widetilde{\gamma}$ at the crease, $\mathcal{F}_1$ to the left, and $\mathcal{F}_2$ to the right. It is from the wikipedia.com page on origami folds. (Ignore the arrows.)

5. Use $f(\mathcal{F}_1)$ and $f_t(\mathcal{F}_2)$ to define a one-parameter family ρ_t of representations $\Sigma \to \mathrm{PO}_0(n,1)$. By letting G_1 act with fundamental domain[3] $\mathcal{F}_1$ and G_2 with fundamental domain $\mathcal{F}_2$. This defines representations $G_j \to \mathrm{SO}_0(2,1) \to \mathrm{PO}_0(n,1)$.
6. By the universal property of amalgamated products, the representations $G_j \to \mathrm{PO}_0(n,1)$, which agree on γ, extend to a unique homomorphism $\rho_t : \Sigma \to \mathrm{PO}_0(n,1)$.

Exercise 10. Prove that, for t sufficiently small, ρ_t is discrete, faithful and, most importantly, *not Fuchsian*. Hint: Show that the Zariski closure of $\rho_t(\Sigma)$ is bigger than $\mathrm{SO}_0(2,1)$.

The important thing to take from this is that Fuchsian representations are highly non-rigid. In other words, one can easily deform Fuchsian representations into non-Fuchsian representations.

Exercise 11. Describe these deformations in terms of conjugation of $\rho(G_2)$ by the centralizer of $\rho(\gamma)$ inside $\mathrm{PO}_0(n,1)$.

2 Complex hyperbolic space

The following is the complexification of the Klein model described above. Also see [7]. Let h be the hermitian form on $\mathbb{C}^{n+1}$ given by

$$h(\alpha_1, \dots, \alpha_{n+1}) = |\alpha_1|^2 + \cdots + |\alpha_n|^2 - |\alpha_{n+1}|^2.$$

The associated hermitian matrix is

$$B = \mathrm{diag}(1, \dots, 1, -1),$$

and we also use B to denote the associated sesquilinear form on $\mathbb{C}^{n+1}$. The *Klein model* of complex hyperbolic space $\mathbf{H}^n_{\mathbb{C}}$ is then

$$\mathcal{CP}^n = \left\{ \text{complex lines } \ell \subset \mathbb{C}^{n+1} \ : \ h|_\ell \text{ is negative} \right\} \subset \mathbb{CP}^n.$$

[3]These are actually convex cores for fundamental domains.

The metric is

$$d(x, y) = 2\cosh^{-1}\left(\frac{B(u, v)}{\sqrt{h(u)h(v)}}\right),$$

where $u, v \in \mathbb{C}^{n+1}$ are vectors representing the lines $x, y \in \mathbb{CP}^n$.

Exercise 12. Take real parts and show that there is a totally geodesic embedding $\mathbf{H}^m \to \mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$ for all $m < n$. Hint: Prove that your map $\mathbf{H}^m \to \mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$ is totally geodesic by showing that it is the fixed point set of an involution.

Exercise 13. Show that $\mathcal{CP}^n$ is holomorphically equivalent to the unit ball in $\mathbb{C}^n$. Conclude that $\mathbf{H}^1_{\mathbb{C}}$ is the Poincaré disk model of $\mathbf{H}^2$.

Exercise 14. Show that the ideal boundary of $\mathcal{CP}^n$ is the set of h-*isotropic* lines, $\{\ell \subset \mathbb{C}^{n+1} \ : \ h|_\ell = 0\}$.

Exercise 15. Show that the (holomorphic) isometry group of $\mathcal{CP}^n$ is exactly $\mathrm{PU}(n, 1)$, the group of projective transformations of $\mathbb{C}^{n+1}$ that fix h. What is the full isometry group?

Exercise 16. For $n = 2$, show that the ideal boundary is the one-point compactification of the 3-dimensional homogeneous space Nil. (Hint: Show that the stabilizer of a point on the boundary has a natural identification with Nil.)

Exercise 17. Show that the totally geodesic subspaces of $\mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$ are either $\mathbf{H}^m$ $(m \leq n)$ or $\mathbf{H}^m_{\mathbb{C}}$ $(m < n)$.

The most important thing to take away from the above is that there are two types of totally geodesic embedding $\mathbf{H}^2 \to \mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$. First are the $\mathbb{R}$-Fuchsian embeddings, which come from 3-dimensional real subspaces of $\mathbb{C}^{n+1}$ on which h defines a quadratic form of signature $(2, 1)$. Second are the $\mathbb{C}$-Fuchsian representations, which come from 2-dimensional complex subspaces of $\mathbb{C}^{n+1}$ on which h has signature $(1, 1)$; these are associated with holomorphic totally geodesic embeddings of the Poincaré disk into $\mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$.

Exercise 18. Show that $\mathbb{R}$-Fuchsian representations of a closed surface group Σ into $\mathrm{Isom}(\mathbf{H}^{\mathrm{n}}_{\mathbb{C}})$ are not rigid for $n \geq 3$. Hint: Use $\mathbf{H}^n$.

Exercise 19. Show that two copies of $\mathbf{H}^1_{\mathbb{C}}$ inside $\mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$ are either disjoint or intersect in a single point. Use this to explain why there is not version of the bending trick from §1 for $\mathbb{C}$-Fuchsian representations. Hint: The intersection is (1) totally geodesic and (2) a complex analytic subspace.

One can also show that $\mathbb{R}$-Fuchsian representations are not rigid when $n = 2$, but that takes some more work [9]. We will see in §4 that $\mathbb{C}$-Fuchsian representations are indeed rigid. In fact, they satisfy extremal properties for the *Toledo invariant*, and are analogous to the Fuchsian representations in Theorem 1.

3 The Gromov Norm

Before we define the Toledo invariant, we need the Gromov norm. This will give us the bounded cohomology class that leads to our replacement for $\mathrm{vol}(\rho)$ in §1. You should know something about this anyways. See [8] for tons more. We begin with the treatment of [3].

Let X be a topological space. For a singular chain $z = \sum a_i \sigma_i \in C_*(X)$, the L^1 norm is given by $\|z\|_1 = \sum |a_i|$. We then have a pseudo-norm on $H_*(X, \mathbb{R})$ (here, a map $H_*(X, \mathbb{R}) \to \mathbb{R}^+ \cup \{0\}$ satisfying the usual properties) by

$$\|x\|_1 = \inf\{\|z\|_1 \ : \ z \text{ a chain representing } x\}.$$

Similarly, if $c \in C^*(X)$ is a singular cochain, we have the sup-norm

$$\|c\|_\infty = \sup\{|c(\sigma)| \ : \ \sigma \text{ a (singular) simplex on } T\}.$$

Then, for a class $\alpha \in H^*(X, \mathbb{R})$, we obtain a pseudo-norm (here, a map $H^*(X, \mathbb{R}) \to \mathbb{R} \cup \{\infty\}$ satisfying the usual properties) by

$$\|\alpha\|_\infty = \inf\{\|c\|_\infty \ : \ c \text{ a cochain representing } \alpha\}.$$

We leave some basic but fundamental properties as exercises.

Exercise 20. For all $x \in H_*(X, \mathbb{R})$ and $\alpha \in H^*(X, \mathbb{R})$,

$$|\alpha(x)| \leq \|x\|_1 \|\alpha\|_\infty.$$

Exercise 21. If $f : X \to Y$ is continuous, then $\|f^*\alpha\|_\infty \leq \|\alpha\|_\infty$ for all $\alpha \in H^*(Y, \mathbb{R})$ and $\|f_* x\|_1 \leq \|x\|_1$ for all $x \in H_*(X, \mathbb{R})$.

Exercise 22. If X is a metric space of nonpositive curvature, we need only consider *geodesic* simplices (i.e., simplices with geodesic edges).

Of fundamental importance for us is the following fact, proved in [8].

Proposition 2. *Let $[S]$ be the fundamental class of a closed Riemann surface S of genus $g \geq 2$. Then $\|[S]\|_1 = 4(g-1) = -2\chi(S)$.*

Proof. Choose a hyperbolic metric on S and represent $[S]$ as $\sum r_i\sigma_i$, where σ_i is a geodesic triangle on S. By Gauss–Bonnet,

$$\sum r_i \operatorname{vol}(\sigma_i) = -2\pi\chi(S).$$

Since all triangles have area at most π, we get

$$2\pi|\chi(S)| \leq \sum |r_i|\pi,$$

so $\|[S]\|_1 \geq 2|\chi(S)|$.

It remains to show that $\|[S]\|_1 \leq 2|\chi(S)|$. Choose the hyperbolic metric associated with the hyperbolic $2g$-gon. We can then triangulate S by $4g$ geodesic triangles, so $\|[S]\|_1 \leq 2|\chi(S)| + 2$. However, we can apply this same triangulation trick to a k-fold covering, which represents $k[S]$, i.e., the pullback to S of the fundamental class of the covering, by $2k|\chi(S)| + 2$ triangles. Thus

$$\|k[S]\|_1 \leq 2k|\chi(S)| + 2$$

which implies that

$$\|[S]\|_1 \leq 2|\chi(S)| + \frac{2}{k}.$$

As $k \to \infty$, we get $\|[S]\|_1 \leq 2|\chi(S)|$, and this proves the proposition. $\square$

4 The Toledo Invariant

For hyperbolic Riemann surfaces, there is a $PO_0(2,1)$-invariant 2-form ω on $\mathbf{H}^2$ that is a *Kähler form*. For any Riemann surface $S = \Gamma\backslash\mathbf{H}^2$, the volume form on S is (up to a constant) the projection $\omega_S \in H^2(S,\mathbb{R})$ of ω to S. Similarly, the metric on $\mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$ is associated with a $\mathrm{PU}(n,1)$-invariant 2-form ω on $\mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$. Given any closed complex hyperbolic n-manifold $M = \Gamma\backslash\mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$, the projection $\omega_M \in H^2(M,\mathbb{R})$ is a nontrivial 2-form such that $\omega_M^n \in H^{2n}(M,\mathbb{R})$ is a multiple of the volume form; in particular, it is nontrivial.

Let $\rho : \Sigma \to \mathrm{PU}(n,1)$ be a representation. We can then define a ρ-equivariant continuous map $f : \mathbf{H}^2 \to \mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$ using the ρ-orbit of a point along with negative curvature. We then define the Toledo invariant

$$\tau(\rho) = \int_S f^*\omega.$$

Exercise 23. Show that $\tau(\rho)$ is independent of the choice of f. Hint: Think about the associated bundle and Exercise 6.

We then have the following fundamental result, due to Toledo.

Theorem 3 (Toledo [10])**.** *Let Σ be a closed surface group of genus $g \geq 2$ and $\rho : \Sigma \to \mathrm{PU}(n,1)$ be a representation. Then*

$$|\tau(\rho)| \leq 2|\chi(S)|.$$

Sketch of proof. We give the proof from [3]. The original proof uses harmonic maps and the so-called Bochner method.

The first step is to prove that $\left|\int_\Delta \omega\right| \leq \pi$ for any triangle Δ in $\mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$. One difficulty worthy of mention is that $\mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$ has variable curvature, so there is a nontrivial moduli space of $\mathrm{PU}(n,1)$-isomorphism classes of triangles. The result is relatively clear when Δ is a geodesic triangle in a holomorphically embedded $\mathbf{H}^2 \subset \mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$. The key is proving that this is the extremal case[4].

It follows that $\|f^*\omega\|_\infty \leq \pi$. From the above properties of the Gromov norm,

$$|\tau(\rho)| = |f^*(\omega)([S])| \leq \|f^*\omega\|_\infty \|S\|_1 \leq 2|\chi(S)|,$$

which proves the theorem. $\square$

Theorem 4 (Toledo [11])**.** *Let Σ be a closed surface group of genus $g \geq 2$ and $\rho : \Sigma \to \mathrm{PU}(n,1)$ a representation. Then $|\tau(\rho)| = 2|\chi(S)|$ if and only if ρ is Fuchsian with respect to a holomorphic totally geodesic embedding $\mathbf{H}^2 = \mathbf{H}^1_{\mathbb{C}} \to \mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$.*

Sketch of proof. The key point is proving that $\rho(\Sigma)$ fixes such a totally geodesic subspace. That ρ is Fuchsian basically follows from Theorem 1 along with properties of the Gromov norm. Using some technical estimates along with the strict negative curvature of $\mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$, one obtains a measurable mapping $d_\rho : \partial\,\mathbf{H}^2 \to \partial\,\mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$. One then proves that if Δ is a (possibly ideal) triangle in $\mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$ with $\int_\Delta \omega = \pi$, then Δ is an ideal triangle in $\mathbf{H}^1_{\mathbb{C}} \subset \mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$. Applying this to arbitrary triples of points on $d_\rho(\partial\,\mathbf{H}^2)$ shows that almost every triple of points lie on the ideal boundary of a fixed $\mathbf{H}^1_{\mathbb{C}} \subset \mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$. This line is $\rho(\Sigma)$-invariant, which gives the theorem. $\square$

[4] For example, if Δ is a triangle in a 'real' totally geodesic $\mathbf{H}^2 \to \mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$, then $\omega|_\Delta \equiv 0$, so the integral is trivial. In a certain sense, all other cases interpolate between these two extremes.

Remark. The basic idea of the proof is related to certain proofs of many rigidity theorems, e.g., Mostow rigidity, in that the important technical step is producing a measurable mapping of ideal boundaries that satisfies nice properties.

5 Generalizations

Over the last ten or fifteen years, there has been a great deal of progress on various generalizations of Toledo's result. First, instead of a Riemann surface, i.e., a complex hyperbolic 1-manifold, we can try to generalize Toledo's results to higher-dimensional complex hyperbolic manifolds. One such result from the same era as Toledo's work was the following rigidity theorem of Goldman–Millson [4].

Theorem 5. *For any $1 < m \leq n$, let $\Gamma < \mathrm{SU}(n-1,1)$ be a torsion free lattice and $\rho : \Gamma \to \mathrm{SU}(n-1,1) \to \mathrm{PU}(n,1)$ be a representation given by factoring the natural representation of Γ into $\mathrm{SU}(n-1,1)$ into $\mathrm{PU}(n,1)$ via a totally geodesic embedding $\mathbf{H}_{\mathbb{C}}^{n-1} \to \mathbf{H}_{\mathbb{C}}^{\mathrm{n}}$. Then Γ is infinitesimally rigid, i.e., any small deformation of Γ also stabilizes a totally geodesic $\mathbf{H}_{\mathbb{C}}^{n-1} \subset \mathbf{H}_{\mathbb{C}}^{\mathrm{n}}$.*

In particular, the 'Fuchsian' representations of Γ form a connected component of $\mathrm{Hom}(\Gamma, \mathrm{PU}(n,1))$. Soon thereafter, Corlette proved the following generalization [2].

Theorem 6. *Let M be a compact complex hyperbolic manifold of (complex) dimension m and fundamental group Γ. If $n > m$ and $\rho : \Gamma \to \mathrm{SU}(n,1)$ is a representation with $\mathrm{vol}(\rho) = \mathrm{vol}(M)$, then $\rho(\Gamma)$ stabilizes a totally geodesic $\mathbf{H}_{\mathbb{C}}^{m} \subset \mathbf{H}_{\mathbb{C}}^{\mathrm{n}}$.*

Here, $\mathrm{vol}(\rho)$ is the pullback of ω^m to $H^{2m}(M, \mathbb{R})$, where ω is the Kähler class on $\mathbf{H}_{\mathbb{C}}^{\mathrm{n}}$, so $\mathrm{vol}(\rho)$ generalizes the Toledo invariant. Since $\mathrm{vol}(\rho)$ varies among a discrete set (e.g., by relation with a characteristic class), this is a strong generalization of the Goldman–Millson result stated above. In particular, the infinitesimal rigidity of Goldman–Millson, analogously to the case of surface group representations, passes to the entire connected component of the 'Fuchsian' representations.

Returning to surface groups, there are two types of generalizations. First, one can allow noncompact surfaces, where similar rigidity results follow once one insists that certain regularity of the $\mathrm{PU}(n,1)$

representation (these are so-called *maximal* representations). Instead, one can replace $\mathbf{H}^n_{\mathbb{C}}$ with some other hermitian symmetric space, i.e., a symmetric space with a 'nice' complex structure. To start down either of these paths, see [1].

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Anosov representations in AdS geometry

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The point of this note is to summarize the work of T. Barbot and Q. Merigot on Anosov representations in anti de Sitter geometry, published as one joint paper [BM], but available on arXiv as two separate papers [Mer] and [Ba1]. Combined, the work of Barbot and Merigot shows that when the target group is $\mathrm{SO}(2,n)$, the notion of Anosov representation (Labourie [La]) coincides exactly with the notion of quasifuchsian: here the natural space on which $\mathrm{SO}(2,n)$ acts is the Einstein universe $\mathrm{Ein}^{n-1,1} = \mathrm{Ein}^n$, a conformal Lorentzian manifold which should be thought of as a Lorentzian analogue of the conformal Riemannian sphere. $\mathrm{SO}(2,n)$ also acts on the $n+1$ dimensional anti de Sitter space $\mathrm{AdS}^{n,1} = \mathrm{AdS}^{n+1}$, which is a Lorentzian model space of constant negative curvature, a Lorentzian analogue of hyperbolic space. The Einstein space $\mathrm{Ein}^{n-1,1}$ is the ideal boundary of $\mathrm{AdS}^{n,1}$, in the same way that the conformal sphere is the boundary of hyperbolic space.

Let Γ be a cocompact lattice is $\mathrm{SO}(n,1)$. Via the inclusion $\mathrm{SO}(n,1) \hookrightarrow \mathrm{SO}(n,2)$ we obtain a *Fuchsian* representation of Γ. This representation preserves a totally geodesic copy of $\mathbb{H}^n$ inside $\mathrm{AdS}^{n,1}$; it also preserves the boundary $\partial\mathbb{H}^n \subset \mathrm{Ein}^{n,1}$, which is a smooth space-like $(n-1)$-sphere in $\mathrm{Ein}^{n-1,1}$. A representation $\rho : \Gamma \to \mathrm{SO}(n,2)$ is *quasi-Fuchsian* if ρ preserves an embedded, *acausal*, $n-1$ sphere Λ (the limit set) in $\mathrm{Ein}^{n-1,1}$. When we say that a representation $\rho : \Gamma \to \mathrm{SO}(n,2)$ is *Anosov*, we will mean Anosov in the context of the action of $\mathrm{SO}(n,2)$ on the subspace $\mathcal{Y} \subset \mathrm{Ein}^n \times \mathrm{Ein}^n$ of pairs of points (v,w) that are *acausal*, meaning there exists a space-like path in (the universal cover) of Ein^n, connecting v to w. The key connection between the notion of Anosov and that of quasi-Fuchsian comes from AdS geometry: The limit set $\Lambda \subset \mathrm{Ein}$ of a quasi-Fuchsian representation ρ is the boundary of a domain in AdS whose quotient under ρ is a *globally hyperbolic Cauchy compact* (GHC) AdS manifold. The geodesic flow on the non-wandering space-like unit tangent bundle in this manifold turns out to be dynamically the same as the geodesic flow on $\Gamma\backslash T^1\mathbb{H}^n$; this allows us to construct the Anosov map from $T^1(\Gamma\backslash\mathbb{H}^n)$ to $\mathcal{Y}$.

Finally, we mention one newer result of Barbot [Ba2], which we will not discuss here: All deformations of the Fuchsian representation of Γ in $\mathrm{SO}(2,n)$ are quasi-Fuchsian. In other words the component of the Fuchsian representation consists entirely of Anosov representations. This is well-known due to Mess in dimension three.

Before starting, a disclaimer: I have left out many of the details (and perhaps even some important ones!). I only intend to give a vague idea of the arguments.

1 Definitions of the basic objects

For background on AdS geometry, see [BB] or my thesis (mostly dimension 3). For a nice introduction to the $2+1$ dimensional Einstein space, see [BCD+]

Consider the quadratic form

$$q_{2,n}(u,v,x_1,\dots,x_n) = -u^2 - v^2 + x_1^2 + \cdots + x_n^2$$

of signature $(n,2)$ on $\mathbb{R}^{n+2} = \mathbb{R}^{2,n}$. Let $\langle \cdot | \cdot \rangle$ denote the associated inner porduct. The anti de Sitter space $\mathrm{AdS}^{n+1} = \mathrm{AdS}^{n,1}$ is defined to be the hyperboloid $\{q_{2,n} = -1\}$ endowed with the metric coming from the restriction of $q_{2,n}$. This metric is *Lorentzian*, meaning that the inner product on each tangent space has signature $(1,n)$ (one negative eigenvalue and n positive eigenvalues).

The Einstein universe $\mathrm{Ein}^{n-1,1} = \mathrm{Ein}^n$ is the positive projectivization of the light cone with respect to $q_{2,n}$:

$$\mathrm{Ein}^{n-1,1} = \{q_{2,n}(u,v,\mathbf{x}) = 0 : \mathbf{x} \neq 0\}/\mathbb{R}^+.$$

It is topologically $\mathbb{S}^1 \times \mathbb{S}^n$; the diffeomorphism is given by

$$(u,v,\mathbf{x}) \mapsto \left(\frac{(u,v)}{\sqrt{u^2+v^2}}, \frac{\mathbf{x}}{\|\mathbf{x}\|} \right),$$

where $\|\cdot\|$ denotes the usual Euclidean norm.

The group $\mathrm{SO}(2,n)$ acts on both $\mathrm{AdS}^{n,1}$ and $\mathrm{Ein}^{n-1,1}$. In the case of AdS, its clear that this action is by isometries. Under the identification $\mathrm{Ein}^{n,1} = \mathbb{S}^1 \times \mathbb{S}^n$ above, the Lorentzian metric $-d\theta^2 + ds^2$ is preserved, *up to conformal factor*, by $\mathrm{SO}(2,n)$. This conformal metric is induced by $q_{2,n}$.

Let $\mathbb{S} : \mathbb{R}^{n+2} \setminus \{0\} \to \mathbb{S}^{n+1}$ denote the quotient by the positive reals $\mathbb{R}^+$. Then $\mathbb{S}$ is injective on both $\mathrm{AdS}^{n,1}$ and $\mathrm{Ein}^{n-1,1}$ and Merigot denotes the images $\mathbb{ADS}^{n,1} = \mathbb{S}(\mathrm{AdS}^{n,1})$ and $\partial\mathbb{ADS}^{n,1} = \mathbb{S}(\mathrm{Ein}^{n-1,1})$. The image of Einstein space is precisely the boundary of the image of AdS. This allows us to think of Ein as an ideal boundary of AdS in the same way that the sphere is the ideal boundary of $\mathbb{H}^n$. The (pushforward of the) metrics work well together: If p_n is a sequence of points in $\mathbb{ADS}$ approaching the point p_∞ in $\partial\mathbb{ADS}$, then the conformal class of the AdS metric at p_n converges to the (conformal class of the) Einstein metric at p_∞. One other important feature that can be seen directly in this model is that every space-like geodesic in $\mathbb{ADS}$ has two distinct end-points on $\partial\mathbb{ADS}$. Let $\mathcal{E}^1\mathrm{AdS}^{n,1}$ denote the unit space-like tangent bundle of AdS, consisting of all tangent vectors with norm-squared $+1$. Then, define the two maps $\ell_+, \ell_- : \mathcal{E}^1\mathrm{AdS}^{n,1} \to \mathrm{Ein}^{n-1,1}$ taking a tangent vector to its forward and backward endpoints (respectively) under the geodesic flow. These will be useful later. Henceforth, we will not distinguish between the two models $\mathbb{ADS}$ and AdS, and we will only use the notation AdS and Ein. Note that some (including me) prefer to take AdS to be the quotient by the antipodal map of the model defined here.

1.1 Causality

Let M be a manifold with Lorentzian metric g, and let $v \neq 0$ be a vector tangent to M at some point. If $g(v,v) < 0$, then v is called *time-like*, if $g(v,v) = 0$ then v is called light-like and if $g(v,v) > 0$, then v is called *space-like*. This terminology comes from physics (Einstein): paths with time-like tangents describe the motion of particles; a geodesic with light-like tangent describes a photon. A hyper-surface with space-like tangents is a copy of space in the space-time, you could think of it as the slice $t = 0$, although in general such a time function t might not extend to the entire space-time M. The notion of *globally hyperbolic* means roughly that there is a global time function t on M. Then, the level sets of t are space-like hypersurfaces.

In any tangent space, the light-like vectors form a cone which divides the time-like vectors into two components. We choose one component to be called *future* and the other to be called *past*. This choice can be made consistently over small patches of the manifold. The manifold is called *time-orientable* if future/past can be chosen consistently over the entire manifold. Even a time-orientable space-time may not have a well-defined notion of future/past for pairs of points. For example, both Ein and AdS contain closed time-like curves, so future/past only makes sense in the universal cover.

Since we wish to deal with rough sets (like limit sets), we need to extend the notion of time-like, space-like, light-like, to curves which are not smooth. Causality notions only make sense in the universal cover; when applied to a manifold which is not simply connected, lift everything to the universal cover before applying the definitions. A curve c is called (future) *causal* if for every $a > b$, $c(a)$ is in the future of $c(b)$, meaning that there exists a smooth curve from $c(a)$ to $c(b)$ with (future directed) time-like or light-like tangents. A set Λ is called *achronal* if for any two points of Λ there is no time-like curve separating those points. A set Λ is *acausal* if for any two points of Λ, there is no causal curve separating those points.

In our context, $\Lambda \subset \mathrm{Ein}^n$ will be the image of an $n-1$-sphere under some injective map. In this case:

Lemma 1. *Λ is acausal if and only if for all $p, q \in \Lambda$, the pair $(p, q) \in \mathcal{Y}$.*

This is not hard to prove. Let $\widetilde{\mathrm{AdS}}^{n,1}$ denote the universal cover, and let $\widetilde{\mathrm{Ein}}^{n-1,1}$ denote the boundary of $\widetilde{\mathrm{AdS}}$ ($n = 2$ is slightly annoying here since $\widetilde{\mathrm{Ein}}$ is not the universal cover of Ein so lets ignore this case here). Let $\widetilde{\Lambda}$ be a lift of Λ to $\widetilde{\mathrm{Ein}}$. If some pair $(p, q) \notin \mathcal{Y}$, then acausal fails: the span of p and q gives a light-like geodesic in Ein^n passing through p and q; this light-like geodesic generates $\pi_1\mathrm{Ein}$ so any lifts $\widetilde{p}, \widetilde{q}$ to $\widetilde{\mathrm{Ein}}$ are connected by the lift of the light-like geodesic. On the other hand, if all pairs of points $p, q \in \Lambda$ satisfy $(p, q) \in \mathcal{Y}$, then we show Λ is acausal as follows. Let $p, q \in \Lambda$. By the assumption $(p, q) \in \mathcal{Y}$, $\mathrm{span}\{p, q\}$ is a $(1, 1)$ plane which descends to a space-like geodesic in $\mathrm{AdS}^{n,1}$. This geodesic is contained in many totally geodesic hyperbolic planes. To find one, pick a positive vector v in $\mathrm{span}\{p, q\}^{\perp}$; then $\mathrm{span}(p, q, v)$ has signature $(1, 2)$ and descends to a hyperbolic plane H in $\mathrm{AdS}^{n,1}$. The boundary ∂H is a space-like curve in Ein^n passing through p and q. Now let α be the loop gotten by traveling from p to q along some path γ inside Λ and then traveling from q back to p along ∂H. Then a homotopy α_t from α to the constant loop can be constructed: α_t is the loop going from p to the point $\gamma(t)$ ($\gamma(0) = p$, $\gamma(1) = q$) along γ and going back to p along $\partial H_{\gamma(t)}$ where $H_\gamma(t)$ is a hyperbolic plane containing the geodesic from p to $\gamma(t)$ (chosen continuously in t; here we need that $(p, \gamma(t)) \in \mathcal{Y}$) for all t). Now lift Λ to $\widetilde{\Lambda} \subset \widetilde{\mathrm{Ein}}$ (Λ is simply connected). Then the space-like curve ∂H lifts to a curve, still spacelike, passing through the lifts of p and q to $\widetilde{\Lambda}$.

2 Anosov representations are quasi-Fuchsian

Let $P \subset \mathrm{SO}(2,n)$ be the stabilizer of an isotropic line. Then, $\mathrm{SO}(2,n)/P = \mathrm{Ein}^n$ is Einstein space. The open orbit $\mathcal{Y} \subset \mathrm{Ein}^n \times \mathrm{Ein}^n$ of the $\mathrm{SO}(2,n)$ action is the subset of pairs of points in Einstein space which are acausally related. These are exactly the pairs of endpoints of spacelike geodesics in AdS^{n+1}. We consider now P-Anosov (Merigot calls them $\mathcal{Y}$-Anosov) representations $\rho : \Gamma \to \mathrm{SO}(2,n)$. Let us recall the definition (Labourie). Consider the unit tangent bundle N of the compact hyperbolic manifold $\Gamma\backslash\mathbb{H}^n$, as well as the flat $\mathcal{Y}$-bundle E_ρ over N associated to ρ:

$$E_\rho = T^1\mathbb{H}^n \times_\rho \mathcal{Y}.$$

Denote the geodesic flow on N by ϕ^t; the flow lifts to E_ρ. Each factor of $\mathcal{Y}$ determines a sub-bundle of TE_ρ: Let E^s denote the span of the tangent directions to the stable sub-manifold of the geodesic flow on N and the tangent directions to the first Ein^n factor (see Spencer's notes). Let E^u denote the span of the tangent directions to the unstable manifold of the geodesic flow on N and the tangent directions to the second Ein^n factor. The tangent bundle of E_ρ splits as

$$TE_\rho = \Delta \oplus E^s \oplus E^u$$

where Δ is the line bundle tangent to the flow ϕ^t.

Then ρ is P-Anosov ($\mathcal{Y}$-Anosov) if there is a ϕ^t-invariant section $s : N \to E_\rho$ and there exists constants $a, b > 0$ such that:

- for any vector v in E^s over a point p of $s(T^1N)$, and for any $t > 0$:

$$\|d_p\phi^t(v)\| \leq be^{-at}\|v\|$$

- for any vector v in E^u over a point p of $s(T^1N)$, and for any $t < 0$:

$$\|d_p\phi^t(v)\| \leq be^{at}\|v\|$$

where $\|\cdot\|$ is some Riemannian metric (since Γ is cocompact, this definition is independent of the metric).

Both Merigot and Barbot use an alternate definition of Anosov (see Merigot Remark 5.4):

Remark 1 (Merigot Remark 5.4)**.** *The Anosov section s is really two ρ-equivariant maps $\ell_\rho^+, \ell_\rho^- : T^1\mathbb{H}^n \to \mathrm{Ein}^n$. An equivalent formulation of the contraction properties is as follows: There exists a family of Riemannian metrics $g^{(x,v)}$ depending continuously on $(x,v) \in T^1\mathbb{H}^n$ and defined in a neighborhood of both $\ell_\rho^+(x,v)$ and $\ell_\rho^-(x,v)$ in Ein^n such that:*

- *The family is ρ-equivariant: Let $w \in T_{\ell_\rho^+(x,v)}$ or $w \in T_{\ell_\rho^-(x,v)}$. Then*

$$g^{\gamma(x,v)}(d\rho(\gamma)w, d\rho(\gamma)w) = g^{(x,v)}(w,w)$$

- *There exists $a, b > 0$, such that if $w \in T_{\ell_\rho^+(x,v)}$ and $t > 0$, then*

$$g^{\phi^t(x,v)}(w,w) \geq b^{-1}\exp(at)g^{(x,v)}(w,w)$$

  or if $w \in T_{\ell_\rho^-(x,v)}$ and $t > 0$ then

$$g^{\phi^t(x,v)}(w,w) \leq b\exp(at)g^{(x,v)}(w,w).$$

We use this second definition from now on, avoiding explicit reference to the metrics $g^{(x,v)}$ whenever the intuition is clear.

We now summarize Merigot's argument showing that Anosov representations (as above) are quasi-Fuchsian. This direction doesn't actually involve very much AdS geometry. In fact the following argument is very general, and is covered in Spencer's notes on the basics of Anosov representations. We repeat it here in this context.

The Anosov section s is really two maps $\ell_\rho^+, \ell_\rho^- : T^1\mathbb{H}^n \to \mathrm{Ein}^n$. Let $(x,v) \in N$ lie on the closed geodesic corresponding to $\gamma \in \Gamma$.

- Both $\ell_\rho^+(x,v), \ell_\rho^-(x,v)$ must be fixed points of $\rho(\gamma)$, and they must be distinct and acausally related. It follows that $\rho(\gamma)$ preserves the space-like geodesic in AdS^{n+1} with endpoints $\ell_\rho^+(x,v), \ell_\rho^-(x,v)$.
- It follows from the previous remark that $\ell_\rho^+(x,v)$ is an *attracting* fixed point of $\rho(\gamma)$ and that $\ell_\rho^-(x,v)$ is a *repelling* fixed point.
- A basic projective geometry argument implies that $\ell_\rho^+(x,v)$ (respectively $\ell_\rho^-(x,v)$) is the unique attracting (respectively repelling) fixed point of $\rho(\gamma)$.

Using these observations, Merigot proves

Proposition 2 (Merigot Prop 5.8)**.** *Let $\alpha : T^1\mathbb{H}^n \to T^1\mathbb{H}^n$ be the map which flips the direction of tangent vectors: $\alpha(x,v) = (x,-v)$. Then $\ell_\rho^+ = \ell_\rho^- \circ \alpha$.*

The statement is true when (x, v) lies on a closed geodesic by the above observations. The proposition then follows by density of closed geodesics.

Now, define $\Lambda = \ell_\rho^+(N) = \ell_\rho^-(N)$. Since ℓ_ρ^+ (or ℓ_ρ^-) is flow invariant, we can really think of it as an equivariant map $\partial\mathbb{H}^n \to \mathrm{Ein}^n$ since the value only depends on the forward endpoint of the geodesic flow of (x, v). Thought of in this way, we argue that ℓ_ρ^+ is injective and its image is acausal. For consider two points (x, v) and (y, w) of $T^1\mathbb{H}^n$ with distinct (forward) endpoints on $\partial\mathbb{H}^n$. There is a third unit tangent vector (z, u) whose forward endpoint is the forward endpoint of (x, v) and whose backwards endpoint is the forward endpoint of (y, w). Then, using the proposition:

$$\begin{aligned}
\ell_\rho^+(x, v) &= \ell_\rho^+(z, u) \\
\ell_\rho^+(y, w) &= \ell_\rho^+(z, -u) = \ell_\rho^-(z, u).
\end{aligned}$$

Since $(\ell_\rho^+(z, u), \ell_\rho^-(z, u)) \in \mathcal{Y}$ it follows that $\ell_\rho^+(x, v)$ and $\ell_\rho^+(y, w)$ are distinct and acausally related. This proves that Λ is an acausal embedded $n - 1$ sphere, and so ρ is quasi-Fuchsian.

3 Globally hyperbolic AdS space-times

Now to the converse result of Barbot, that quasi-Fuchsian representations are $\mathcal{Y}$-Anosov. We give only a vague summary. The strategy involves the geometry of globally hyperbolic AdS space-times.

Consider an acausal subset Λ of $\mathrm{Ein}^{n-1,1}$. Let $E(\Lambda)$ denote the points of $\mathrm{AdS}^{n,1}$ that can be joined to all points of Λ by a space-like geodesic. $E(\Lambda)$ is called the invisible set. It is open. Now, suppose $\rho : \Gamma \to \mathrm{SO}(2, n)$ is quasi-Fuchsian and Λ is the acausal $n - 1$ sphere preserved by the image of ρ. Then $M = \rho(\Gamma)\backslash E(\Lambda)$ is a globally hyperbolic maximal compact (GHMC) AdS manifold: It is globally hyperbolic with compact space-like level sets, and it is maximal with respect to inclusion (Mess). I won't get into the details here, but the assosiation $(\rho, \Lambda) \mapsto M$ is a bijection (Mess [Mes]).

3.1 The unit tangent bundle

Consider the space like unit tangent bundle $\mathcal{E}^1E(\Lambda)$. Define the non-wandering set $\mathcal{N}(\Lambda)$ to be the subset of points (x, v) of $\mathcal{E}^1E(\Lambda)$ such that the space-like geodesic passing though x tangent to v has both endpoints in Λ; in other words $\ell_+(x, v), \ell_-(x, v) \in \Lambda$. The non-wandering set is of course ρ invariant. The basic idea is that $\mathcal{N}(\Lambda)$ gives the connection between the dynamics of the geodesic flow on $\Gamma\backslash\mathbb{H}^n$ and the action of $\rho(\Gamma)$ on Λ. Specifically, Barbot proves:

Proposition 3 (Barbot Proposition 4.19)**.** *There is a ρ-equivairant homeomorphism $\mathfrak{f} : T^1\mathbb{H}^n \to \mathcal{N}(\Lambda)$ mapping orbits of the geodesic flow ϕ^t on $T^1\mathbb{H}^n$ to orbits of the space-like geodesic flow $\phi^t_{\mathcal{N}}$ on $\mathcal{N}(\Lambda)$.*

To prove this, the first step is to show the existence of a ρ-equivariant homeomorphism $j : \partial\mathbb{H}^n \to \Lambda$. This involves some AdS geometry: the group Γ is quasi-isometric to the convex hull $\mathrm{Conv}(\Lambda)$ of Λ in $\mathrm{AdS}^{n,1}$. Then both $T^1\mathbb{H}^n$ and $\mathcal{N}(\Lambda)$ are $\mathbb{R}$-bundles over $\partial\mathbb{H}^n \times \partial\mathbb{H}^n \setminus \mathcal{D} \cong \Lambda \times \Lambda \setminus \mathcal{D}$ (where $\mathcal{D}$ is the diagonal) and $j \times j$ gives an equivalence of the orbit spaces. Construction of $\mathfrak{f}$ is a standard exercise.

From the proposition, we get two flow invariant maps $\ell^+_\rho, \ell^-_\rho : T^1\mathbb{H}^n \to \Lambda$ defined by first applying $\mathfrak{f}$ and then following the geodesic flow in $\mathcal{N}(\Lambda)$ forwards and backwards to Λ. These are of course ρ-equivariant. The focus now is demonstrate the contraction properties of Remark 1 above. The metric $g^{(x,v)}$ at (near) $\ell^+_\rho(x, v)$ is defined as the visual metric from the point $x \in \mathrm{AdS}^{n,1}$. To make it Riemannian, we use *Wick rotation*. Specifically, there is a ρ-equivariant time-like vector field V defined in the convex core $\mathrm{Conv}(\Lambda)$ (which is the projection of $\mathcal{N}(\Lambda)$ to AdS); One may define a Riemannian metric on $\mathrm{Conv}(\Lambda)$ by using the AdS metric on the orthogonal to V and then flipping the sign of $\langle V, V\rangle$; projecting this metric (at x) onto the boundary Ein^n gives a Riemannian metric in a neighborhood of $\ell^+_\rho(x, v)$ (in fact defined on the entire space-like visual "sphere" of x, a copy of de Sitter space dS^n in Ein^n centered around x). Some fiddling is needed to make sure this is equivariant. Barbot shows that this metric satisfies the contracting properties. A key step is to show that for a sequence of group elements γ_n converging to a point in the Gromov boundary $\partial\Gamma$, the elements $\rho(\gamma_n) \in \mathrm{SO}(2, n)$ always have un-balanced distortion, meaning (most of) Einstein space is pushed toward one attracting fixed point.

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CONVEX COCOMPACT GROUPS IN REAL RANK ONE AND HIGHER

FANNY KASSEL

Notes for a talk at the GEAR workshop on Higher Teichmüller–Thurston Theory, Northport, Maine, 23–30 June 2013

Let G be a real, connected, noncompact, semisimple, linear Lie group. The goal of this talk is to recall the classical notion of a convex cocompact subgroup of G when G has real rank 1 and to discuss possible generalizations to higher real rank, following Kleiner–Leeb [KL2] and Quint [Q].

Acknowledgements. I would like to thank the GEAR Network and the organizers of the workshop for a very enjoyable and productive week.

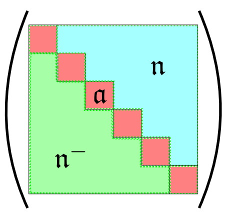
1. Preliminary notation

In the whole talk, K denotes a maximal compact subgroup of G, so that $X = G/K$ is the Riemannian symmetric space of G, and A denotes a maximal split connected semisimple abelian subgroup of G such that the Cartan decomposition $G = KAK$ holds; by definition, the real rank of G is $\mathrm{rank}_{\mathbb{R}}(G) = \dim(A)$. Consider the decomposition of the Lie algebra $\mathfrak{g}$ of G into joint eigenspaces for the adjoint action of A:

$$\mathfrak{g} = (\mathfrak{m} + \mathfrak{a}) + \mathfrak{n} + \mathfrak{n}^-, \tag{1.1}$$

where $\mathfrak{a}$ is the Lie algebra of A, where $\mathfrak{m}$ is the Lie algebra of the centralizer of A in K, and where $\mathfrak{n}$ (resp. $\mathfrak{n}^-$) is the sum of the positive (resp. negative) root spaces with respect to some fixed choice of positive restricted root system. Let M be the centralizer of A in K, with Lie algebra $\mathfrak{m}$, and let N (resp. N^-) be the image of $\mathfrak{n}$ (resp. $\mathfrak{n}^-$) under the exponential map $\exp : \mathfrak{g} \to G$. The group MA is the centralizer $Z_G(A)$ of A in G. The groups $P := (MA) \ltimes N$ and $P^- := (MA) \ltimes N^-$ are opposite minimal parabolic subgroups of G, with respective unipotent radicals N and N^-.

Example 1.1. For $G = \mathrm{SL}_m(\mathbb{R})$, we may take K to be $\mathrm{SO}(m)$ and A to be the group of diagonal matrices in G with positive entries. The Lie algebra $\mathfrak{g}$ is the set of traceless $m \times m$ real matrices, $\mathfrak{a}$ is the subset of diagonal matrices, and $\mathfrak{m}$ is trivial (G is split). With the usual choice of positive root system, $\mathfrak{n}$ (resp. $\mathfrak{n}^-$) is the set of upper (resp. lower) triangular nilpotent matrices.



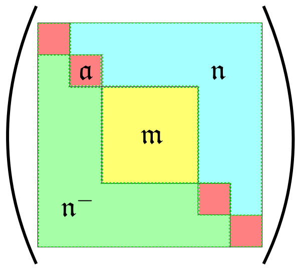
Example 1.2. For $G = \mathrm{SO}(p,q)_0$, the group K is isomorphic to $\mathrm{SO}(p)\times\mathrm{SO}(q)$. Suppose $p \leq q$. If we see G as the subgroup of $\mathrm{SL}_{p+q}(\mathbb{R})$ preserving the quadratic form

$$2\,x_1\,x_{p+q} + 2\,x_2\,x_{p+q-1} + \cdots + 2\,x_p\,x_{q+1} - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_q^2,$$

then we may take $\mathfrak{a}$ to be the set of diagonal matrices

$$\mathrm{diag}(t_1, \ldots, t_p, 0, \ldots, 0, -t_p, \ldots, -t_1)$$

with $t_1, \ldots, t_p \in \mathbb{R}$. In this case $\mathfrak{m}$ is the set of block diagonal matrices $\mathrm{diag}(0, D, 0)$ with $D \in \mathfrak{so}(q-p)$, with the two 0 blocks of size $p \times p$. With the usual choice of positive restricted root system, $\mathfrak{n}$ (resp. $\mathfrak{n}^-$) consists of strictly upper (resp. lower) triangular block matrices.



2. Convex cocompactness in real rank 1

2.1. **Definitions, examples, and characterizations.** In this section we assume that G has real rank 1. Recall the following classical definition.

Definition 2.1 ($\mathrm{rank}_{\mathbb{R}}(G) = 1$)**.** A discrete subgroup Γ of G (or any injective representation $\rho : \Gamma_0 \to G$ of a discrete group Γ_0 into G with image Γ) is said to be *convex cocompact* if one of the following equivalent conditions holds:

(1) Γ preserves and acts cocompactly on some nonempty convex subset $\mathcal{C}$ of $X = G/K$;
(2) Γ acts cocompactly on the closure F_Γ of the union of the translation axes of the hyperbolic elements of Γ in X (in other words, the closure of the union of all closed geodesics is compact in $\Gamma\backslash X$);
(3) the conservative set of the geodesic flow on the unit tangent bundle $T^1(\Gamma\backslash X)$ is compact.

By definition, the conservative set of the geodesic flow $(\varphi_t)_{t\in\mathbb{R}}$ is the closure of the set of vectors $v \in T^1(\Gamma\backslash X)$ such that $\varphi_t(v)$ returns to a compact subset of $T^1(\Gamma\backslash X)$ infinitely often.

Since $\mathrm{rank}_{\mathbb{R}}(G) = 1$, the Riemannian symmetric space X is Gromov-hyperbolic; its visual boundary identifies with G/P. Any nonempty, Γ-invariant, closed, convex subset of X must contain the convex hull $\mathrm{Conv}(\Lambda_\Gamma) \subset X$ of the limit set $\Lambda_\Gamma \subset G/P$ of Γ, hence (1) is equivalent to the fact that Γ acts cocompactly on $\mathrm{Conv}(\Lambda_\Gamma)$. The set F_Γ is the union of all geodesics of X with both endpoints in Λ_Γ; it is contained in $\mathrm{Conv}(\Lambda_\Gamma)$, so (1) immediately implies (2). The following classical lemma (see [Gd, Lem. 4] for instance) explains why (2) implies (1).

Lemma 2.2. *The set* $\mathrm{Conv}(\Lambda_\Gamma)$ *is contained in a uniform neighborhood of its subset* F_Γ.

Convex cocompact groups in real rank 1 include uniform lattices and Schottky groups (with no parabolic element). Other interesting examples are given by quasi-Fuchsian representations of surface groups into $\mathrm{PSL}_2(\mathbb{C})$.

We shall recall a proof of the following well-known lemma.

Lemma 2.3. *If* $\mathrm{rank}_{\mathbb{R}}(G) = 1$*, then the following conditions are equivalent:*

- *(i)* Γ *is convex cocompact,*
- *(ii)* Γ *is finitely generated and quasi-isometrically embedded in* G*,*
- *(iii)* Γ *is Gromov-hyperbolic and there exists a continuous, injective, and* Γ*-equivariant map* $\xi : \partial_\infty\Gamma \to G/P = \partial_\infty X$*,*
- *(iv)* Γ *is Gromov-hyperbolic and the natural inclusion of* Γ *in* G *is a* P*-Anosov representation.*

Recall that "Γ quasi-isometrically embedded in G" means that for some (hence any) finite generating subset S of Γ and some (hence any) G-invariant Riemannian metric d_G on G, there exist $c, C > 0$ such that for any $\gamma \in \Gamma$,

$$d_G(e, \gamma) \geq c\,\ell_S(\gamma) - C,$$

where $\ell_S : \Gamma \to \mathbb{N}$ is the word length function with respect to S. We shall recall the definition of Anosov representations in Section 2.3 below. The implication $(iv) \Rightarrow (ii)$ actually holds when G has arbitrary real rank [GW, Th. 5.3].

A proof of Lemma 2.3 will be given in Section 2.4. We first recall an interpretation of the homogeneous space G/MA.

2.2. **The homogeneous space G/MA.** When $\mathrm{rank}_{\mathbb{R}}(G) = 1$, there are two G-orbits in $G/P \times G/P$: a closed one, namely

$$\mathrm{Diag}(G/P \times G/P) = \{(\xi, \xi) : \xi \in G/P\},$$

and an open one, namely

$$(G/P)^{(2)} = (G/P \times G/P) \smallsetminus \mathrm{Diag}(G/P \times G/P),$$

which we shall denote by $\mathcal{O}$. If we see G/P as the space of minimal parabolic subgroups of G, with the transitive action of G by conjugation, then $(P, P^-) \in G/P \times G/P$ belongs to $\mathcal{O}$; its stabilizer is $P \cap P^- = MA$, hence $\mathcal{O}$ identifies with G/MA. We can interpret $\mathcal{O}$ as the space of oriented geodesics of $X = G/K$, with the two projections of $\mathcal{O} \subset G/P \times G/P$ onto G/P giving the two endpoints in $\partial_\infty X$. This is known as the *Hopf parameterization* of the space of oriented geodesics of X.

When G has arbitrary real rank, G/MA still identifies with the only open G-orbit in $G/P \times G/P$; there may be more than two G-orbits.

2.3. **Anosov representations.** In this section G can have any real rank.

2.3.1. *Exponential expansion and contraction.* We shall use the following classical terminology. Let $\pi : \mathcal{V} \to \mathcal{M}$ be a vector bundle over a compact manifold $\mathcal{M}$. For any $m \in \mathcal{M}$, choose a norm $\|\cdot\|_m$ on the fiber $\pi^{-1}(m)$ so that the family $(\|\cdot\|_m)_{m\in\mathcal{M}}$ is continuous.

Definition 2.4. A flow $(f_t)_{t\in\mathbb{R}}$ on $\mathcal{V}$ is said to *exponentially expand* (resp. *contract*) $\mathcal{V}$ if there exist $c, C > 0$ such that

$$\|f_t \cdot v\|_{\pi(f_t\cdot v)} \geq C\, e^{c|t|}\, \|v\|_{\pi(v)}$$

for all $t \geq 0$ (resp. $t \leq 0$) and all $v \in \mathcal{V}$.

This does not depend on the choice of the continuous family $(\|\cdot\|_m)_{m\in\mathcal{M}}$, by compactness of $\mathcal{M}$.

2.3.2. *Anosov representations for fundamental groups of closed, negatively curved manifolds.* Let $\mathcal{N}$ be a closed, negatively curved, Riemannian manifold, with universal covering $\widetilde{\mathcal{N}}$ and fundamental group Γ_0. The geodesic flow $(\varphi_t)_{t\in\mathbb{R}}$ on the unit tangent bundle $\mathcal{M} := T^1(\mathcal{N})$ is an *Anosov flow*: there exists a $(d\varphi_t)_{t\in\mathbb{R}}$-invariant splitting

$$T\mathcal{M} = \mathscr{D}^0 \oplus \mathscr{D}^+ \oplus \mathscr{D}^- \tag{2.1}$$

of the tangent bundle $T\mathcal{M}$ into subbundles such that $\mathscr{D}^0$ is tangent to the flow and $\mathscr{D}^+$ (resp. $\mathscr{D}^-$) is exponentially expanded (resp. exponentially contracted) by $(d\varphi_t)_{t\in\mathbb{R}}$ in the sense of Definition 2.4. The notion of Anosov representation aims to generalize this idea to representations of Γ_0 into G. In order to define it, we first make a few remarks. Let $\rho : \Gamma_0 \to G$ be a group homomorphism.

• The group Γ_0 acts on $\widehat{\mathcal{M}} := T^1(\widetilde{\mathcal{N}})$ by deck transformations. We denote by $\widehat{\mathcal{M}} \times_\rho G/MA$ the quotient of $\widehat{\mathcal{M}} \times G/MA$ by the diagonal action

$$\gamma \cdot (\hat{m}, gMA) = \big(\gamma \cdot \hat{m},\, \rho(\gamma)gMA\big)$$

of Γ_0. It is a principal bundle over $\mathcal{M}$ with fiber G/MA.

• The geodesic flow on $\widehat{\mathcal{M}}$, which we also denote by $(\varphi_t)_{t\in\mathbb{R}}$, lifts to a flow $(\psi_t)_{t\in\mathbb{R}}$ on $\widehat{\mathcal{M}} \times G/MA$ which is trivial on the right factor:

$$\psi_t \cdot (\hat{m},\, gMA) = (\varphi_t \cdot \hat{m},\, gMA).$$

Since $(\varphi_t)_{t\in\mathbb{R}}$ commutes with the action of Γ_0 on $\widehat{\mathcal{M}}$, the flow $(\psi_t)_{t\in\mathbb{R}}$ descends to a flow on $\widehat{\mathcal{M}} \times_\rho G/MA$, still denoted $(\psi_t)_{t\in\mathbb{R}}$, which lifts $(\varphi_t)_{t\in\mathbb{R}}$.

• Recall from Section 2.2 that the homogeneous space G/MA identifies with an open subset of $G/P \times G/P$. This product structure induces a G-invariant splitting

$$T(G/MA) = \mathscr{E}^+ \oplus \mathscr{E}^-.$$

In turn, the G-invariant distributions $\mathscr{E}^\pm$ on G/MA determine distributions $\mathscr{F}^\pm \subset T(\widehat{\mathcal{M}} \times_\rho G/MA)$ on $\widehat{\mathcal{M}} \times_\rho G/MA$, which are preserved by the flow $(d\psi_t)_{t\in\mathbb{R}}$.

Definition 2.5 (Labourie [L]). A representation $\rho \in \mathrm{Hom}(\Gamma_0, G)$ is *P-Anosov* if there exists a continuous section s of the bundle $\widehat{\mathcal{M}} \times_\rho G/MA \to \mathcal{M}$ which is compatible with the flows $(\varphi_t)_{t\in\mathbb{R}}$ and $(\psi_t)_{t\in\mathbb{R}}$, in the sense that $s\circ\varphi_t = \psi_t\circ s$ for all $t \in \mathbb{R}$, and such that $s^*\mathscr{F}^+$ (resp. $s^*\mathscr{F}^-$) is exponentially expanded (resp. contracted) by $(d\psi_t)_{t\in\mathbb{R}}$.

Another way to say that s is compatible with the flows $(\varphi_t)_{t\in\mathbb{R}}$ and $(\psi_t)_{t\in\mathbb{R}}$ is to say that it is flat along the flow lines of $(\varphi_t)_{t\in\mathbb{R}}$, for the flat connection on the bundle $\widehat{\mathcal{M}} \times_\rho G/MA \to \mathcal{M}$ induced by the flat connection coming from the product structure on $\widehat{\mathcal{M}} \times G/MA \to \widehat{M}$.

2.3.3. *An equivalent definition.* A section s of the bundle $\widehat{\mathcal{M}}\times_\rho G/MA \to \mathcal{M}$ determines a unique ρ-equivariant map $\sigma : \widehat{\mathcal{M}} \to G/MA$, and vice versa. Therefore, Definition 2.5 can be reformulated as follows: a representation $\rho \in \mathrm{Hom}(\Gamma_0, G)$ is P-Anosov if and only if there exists a continuous, ρ-equivariant map $\sigma : \widehat{\mathcal{M}} \to G/MA$ with the following properties:

- σ is invariant under $(\varphi_t)_{t\in\mathbb{R}}$,
- for some (hence any) continuous, ρ-equivariant family $(\|\cdot\|_{\hat{m}})_{\hat{m}\in\widehat{\mathcal{M}}}$, where $\|\cdot\|_{\hat{m}}$ is a norm on $T_{\sigma(\hat{m})}(G/MA)$, there exist $c, C > 0$ such that

$$\|w\|_{\varphi_t\cdot\hat{m}} \geq C\, e^{c|t|}\, \|w\|_{\hat{m}} \tag{2.2}$$

for all $\hat{m} \in \widehat{\mathcal{M}}$ and all $(w,t) \in (\mathscr{E}^+_{\sigma(\hat{m})} \times \mathbb{R}^+) \cup (\mathscr{E}^-_{\sigma(\hat{m})} \times \mathbb{R}^-)$.

In this case, σ factors through the space $\widehat{\mathcal{M}}/(\varphi_t)_{t\in\mathbb{R}}$ of oriented geodesics of $\widetilde{\mathcal{N}}$ and is unique: any $\gamma \in \Gamma_0 \smallsetminus \{e\}$ has a unique attracting (resp. repelling) fixed point ξ^+_γ (resp. ξ^-_γ) in G/P, and σ must send the oriented translation axis of γ in $\widetilde{N}$ to $(\xi^+_\gamma, \xi^-_\gamma) \in G/MA$.

2.3.4. *A characterization in terms of roots.* Here is an equivalent formulation of (2.2) due to Guichard–Wienhard [GW], which we give in the special case where $\mathrm{rank}_{\mathbb{R}}(G) = 1$. Since A is contractible, by using a partition of unity we can lift the continuous, ρ-equivariant map $\sigma : \widehat{\mathcal{M}} \to G/MA$ to a continuous, ρ-equivariant map $\sigma' : \widehat{\mathcal{M}} \to G/M$; any two such lifts differ by a bounded amount in any G-invariant Riemannian metric on G/M. Note that A acts on G/M by right multiplication (because M and A commute). For any $\hat{m} \in \widehat{\mathcal{M}}$, there exists a continuous map $t \mapsto a_{\hat{m},t} \in A$ such that

$$\sigma'(\varphi_t \cdot \hat{m}) = \sigma'(\hat{m})\, a_{\hat{m},t}$$

for all $t \in \mathbb{R}$. The tangent space $T(G/M)$ admits a natural right-A-invariant splitting

$$T(G/M) = \mathscr{G}^0 \oplus \mathscr{G}^+ \oplus \mathscr{G}^-$$

into subbundles such that the image of $\mathscr{G}^0$ (resp. $\mathscr{G}^+$, resp. $\mathscr{G}^-$) under the projection $T(G/M) \to T(G/MA)$ is $\{0\}$ (resp. $\mathscr{E}^+$, resp. $\mathscr{E}^-$). To simplify, suppose that the rank-one group G has only one positive restricted root α (in general, when G is not split, it can have two positive roots α and 2α). Then for any $a \in A$, the differential map $T(G/M) \to T(G/M)$ of the right multiplication by a restricts to the trivial map on $\mathscr{G}^0$, to the multiplication by the scalar $\alpha(a) > 0$ on $\mathscr{G}^+$, and to the multiplication by $1/\alpha(a)$ on $\mathscr{G}^-$. From this observation (and its variant when G has two positive roots α and 2α), one easily deduces that (2.2) is equivalent to the existence of $c, C > 0$ such that

$$\alpha(a_{\hat{m},t}) \geq C\, e^{ct} \tag{2.3}$$

for all $(\hat{m},t)\in\widehat{\mathcal{M}}\times\mathbb{R}_+$. Note that if $\hat{m}$ belongs to a periodic orbit of length T of the geodesic flow $(\varphi_t)_{t\in\mathbb{R}}$, corresponding to an element $\gamma\in\Gamma_0$, then

$$\sigma'(\gamma\cdot\hat{m})=\rho(\gamma)\cdot\sigma'(\hat{m})=\sigma'(\hat{m})\,a_{\hat{m},T},$$

hence $\rho(\gamma)$ is conjugate in G to an element of $Ma_{\hat{m},T}\subset MA$ and $\log\alpha(a_{\hat{m},T})$ is the length in $\rho(\Gamma)\backslash X$ of the closed orbit corresponding to $\rho(\gamma)$. A formulation of (2.2) similar to (2.3), but involving more than one root α, holds when G has arbitrary real rank (see [GW, § 3.3]).

2.3.5. *Anosov representations for general Gromov-hyperbolic groups.* The definition of Anosov representation has been extended in [GW] to any Gromov-hyperbolic group Γ_0: by [Gr, M], there is a natural action of the group $\Gamma_0\times\mathbb{R}\rtimes(\mathbb{Z}/2\mathbb{Z})$ on some proper hyperbolic metric space $\widehat{\Gamma}_0$ (the *flow space*, replacing $\widehat{\mathcal{M}}=T^1(\widetilde{\mathcal{N}})$ in the situation above) such that

- the action of $\Gamma_0\times(\mathbb{Z}/2\mathbb{Z})$ on $\widehat{\Gamma}_0$ is isometric;
- every orbit map $\Gamma_0\to\widehat{\Gamma}_0$ is a quasi-isometry; in particular, $\partial_\infty\Gamma_0\simeq\partial_\infty\widehat{\Gamma}_0$;
- the action of $\mathbb{R}$ on $\widehat{\Gamma}_0$ is free, and every orbit map $\mathbb{R}\to\widehat{\Gamma}_0$ is a quasi-isometric embedding; the induced map $\widehat{\Gamma}_0/\mathbb{R}\to(\partial_\infty\widehat{\Gamma}_0)^{(2)}$ is a homeomorphism.

A representation $\rho:\Gamma_0\to G$ is said to be *P-Anosov* if there exists a continuous, ρ-equivariant, $\mathbb{R}$-invariant map $\sigma:\widehat{\Gamma}_0\to G/MA$ and, for any continuous, ρ-equivariant family $(\|\cdot\|_{\hat{m}})_{\hat{m}\in\widehat{\Gamma}_0}$ of norms on $T_{\sigma(\hat{m})}(G/MA)$, constants $c,C>0$ such that (2.2) holds for all $\hat{m}\in\widehat{\Gamma}_0$ and all (w,t) in $\mathscr{E}^+_{\sigma(\hat{m})}\times\mathbb{R}^+$ or $\mathscr{E}^-_{\sigma(\hat{m})}\times\mathbb{R}^-$. As above, this is equivalent to the existence of $c,C>0$ such that (2.3) holds for all $(\hat{m},t)\in\widehat{\Gamma}_0\times\mathbb{R}_+$.

2.4. **Proof of Lemma 2.3.** The implication $(i)\Rightarrow(ii)$ is immediate: if Γ is convex cocompact, then it acts cocompactly on $\mathrm{Conv}(\Lambda_\Gamma)\subset X$, hence it is finitely generated and any orbit map $\Gamma\to X$ is a quasi-isometric embedding. On the other hand, any orbit map $G\to X$ is a quasi-isometry since K is compact.

To establish $(ii)\Rightarrow(i)$, it is sufficient to see that if Γ is finitely generated and quasi-isometrically embedded in G, then for any $x\in X$ the set F_Γ of Definition 2.1.(2) is contained in a uniform neighborhood of the Γ-orbit of x. Choose a finite generating subset S of Γ and let

$$r=\max_{s\in S}d_X(x,s\cdot x)>0.$$

Quasi-isometricity implies the existence of constants $c,C>0$ such that for any $\gamma,\gamma'\in\Gamma$, there is a (c,C)-quasi-geodesic $\mathcal{G}_{\gamma,\gamma'}$ from $\gamma\cdot x$ to $\gamma'\cdot x$ which is a concatenation of geodesic segments of length $\leq r$ with endpoints in $\Gamma\cdot x$. Since X is Gromov-hyperbolic, there is a constant $R>0$ (depending only on (c,C) and on the hyperbolicity constant of X) such that any point of the geodesic segment $[\gamma\cdot x,\gamma'\cdot x]$ lies at distance $\leq R$ from $\mathcal{G}_{\gamma,\gamma'}$, hence at distance $\leq R+r$ from $\Gamma\cdot x$. Since Λ_Γ is the set of accumulation points of $\Gamma\cdot x$, we obtain that F_Γ remains at distance $\leq R+r$ from $\Gamma\cdot x$ by taking a limit.

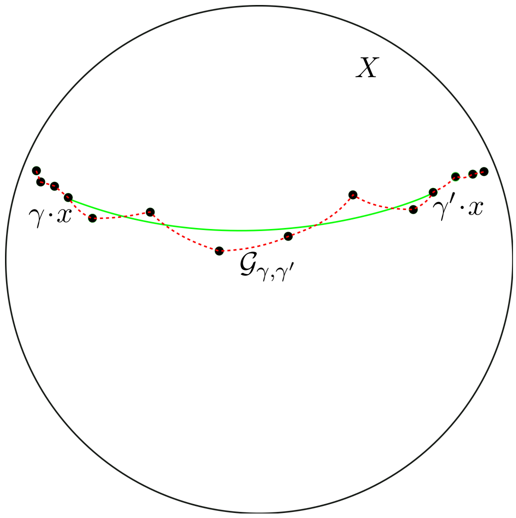


FIGURE 1. If Γ is quasi-isometrically embedded in G, then all geodesics of X with both endpoints in the limit set Λ_Γ are contained in some uniform neighborhood of $\Gamma \cdot x$.

For $(ii) \Rightarrow (iii)$, note that if the natural inclusion of Γ in G is a quasi-isometric embedding, then so is any orbit map $\Gamma \to X$ by compactness of K. In particular, Γ is Gromov-hyperbolic and any orbit map induces a continuous, injective, Γ-equivariant map $\xi : \partial_\infty \Gamma \to \partial_\infty X = G/P$.

To establish $(iii) \Rightarrow (iv)$, we can use the natural homeomorphisms $\widehat{\Gamma}/\mathbb{R} \simeq (\partial_\infty\Gamma)^{(2)}$ and $(G/P)^{(2)} \simeq G/MA$ from Sections 2.2 and 2.3 to obtain, from ξ, a continuous, Γ-equivariant, $\mathbb{R}$-invariant map

$$\sigma : \widehat{\Gamma} \longrightarrow G/MA.$$

Let $\sigma' : \widehat{\Gamma} \to G/M$ be a continuous, Γ-equivariant lift of σ. By Section 2.3, it is sufficient to prove the existence of constants $c, C > 0$ such that (2.3) holds for all $\hat{m} \in \widehat{\Gamma}$ and $t \geq 0$. This follows from the compactness of $\Gamma \backslash \widehat{\Gamma}$: see [GW, Th. 4.11] for details in the case that Γ is Zariski-dense in G. (We can reduce to the Zariski-dense case by using the general fact [Bo] that a Zariski-closed subgroup of G is either reductive or contained in a proper parabolic subgroup of G.)

The implication $(iv) \Rightarrow (ii)$ is true when G has arbitrary real rank; we explain it for $\mathrm{rank}_{\mathbb{R}}(G) = 1$. (The general proof is similar, using a version of (2.3) involving more than one root α.) Recall from Section 2.3 that if (iv) holds, then there exist a continuous, Γ-equivariant, $\mathbb{R}$-invariant map $\sigma : \widehat{\Gamma} \to G/MA$ and, for any continuous, Γ-equivariant lift $\sigma' : \widehat{\Gamma} \to G/M$ of σ, constants $c, C > 0$ such that (2.3) holds for all $\hat{m} \in \widehat{\Gamma}$ and $t \geq 0$. The condition (2.3) implies that the restriction of σ' to any periodic $\mathbb{R}$-orbit is a (c, C')-quasi-isometric embedding, for some $C' > 0$ independent of the $\mathbb{R}$-orbit. On the other hand, the hyperbolicity of $\widehat{\Gamma}$ implies the existence of a constant $R > 0$ with the following property: for any $y, y' \in \widehat{\Gamma}$, there exist $z, z' \in \widehat{\Gamma}$ such that z and z' belong to the same $\mathbb{R}$-orbit and the distances in $\widehat{\Gamma}$ between y and z, and between y' and z', are $\leq R$. Therefore, $\sigma' : \widehat{\Gamma} \to G/M$

is a quasi-isometric embedding. Since M is compact, Γ is quasi-isometrically embedded in G.

3. Generalizations of convex cocompactness to higher rank

We now allow the semisimple group G to have arbitrary real rank. Let G_1 (resp. G_2) be the product of the simple factors of G of real rank 1 (resp. of real rank ≥ 2). (Products are in fact almost products, but this is not very important.) The Riemannian symmetric space $X = G/K$ is the product of the Riemannian symmetric spaces X_1 of G_1 and X_2 of G_2.

Fix a discrete subgroup Γ of G. If H is the Zariski-closure of Γ in G and $R(H)$ its radical, then the group $H/R(H)$ is semisimple and the image of Γ in $H/R(H)$ is discrete (see [R, Cor. 8.27]). We may therefore restrict to the case where Γ is Zariski-dense in G, up to replacing G with a smaller group.

3.1. Generalization of (1).

In answer to a question of Corlette from 1994, Kleiner and Leeb proved that in higher real rank, Definition 2.1.(1) for convex cocompactness does not give any interesting example of discrete groups apart from products of uniform lattices and convex cocompact subgroups of rank-one factors.

Theorem 3.1 (Kleiner–Leeb [KL2])**.** *Let $\mathcal{C} \neq \emptyset$ be a Γ-invariant, closed, convex subset of X with $\Gamma\backslash\mathcal{C}$ compact. If Γ is Zariski-dense in G, then $\mathcal{C} = \mathcal{C}_1 \times X_2$ for some Γ-invariant, closed, convex subset $\mathcal{C}_1$ of X_1; in particular, Γ is a product of convex cocompact subgroups of the rank-1 factors of G_1 and of a uniform lattice of G_2.*

The proof easily reduces to the case that G is simple of real rank ≥ 2. In this case, Kleiner and Leeb use results from [Be] and [KL1] to prove that the geometric boundary of $\mathcal{C}$ is a top-dimensional subbuilding of the spherical building at infinity (Tits boundary) of X, which is a closed subset with respect to the topology of the visual boundary of X. The main step in [KL2] is then to show that any such subbuilding is either equal to the full visual boundary of X or contained in the visual boundary of a proper symmetric subspace of X. This last possibility is ruled out by the Zariski-density assumption on Γ, and so $\mathcal{C} = X$.

On the other hand, Quint [Q] investigated generalizations of the definitions (2) and (3) of convex cocompactness from Section 2.1. Before we explain his work, we recall interpretations of the homogeneous spaces G/P, G/MA, and G/M in arbitrary real rank.

3.2. The homogeneous spaces G/P, G/MA, and G/M.

In real rank 1, the homogeneous space G/P identifies with the visual boundary of $X = G/K$, namely with the set of equivalence classes of geodesic rays of X for the relation "to be at finite Hausdorff distance". In higher real rank, the visual boundary of X is more complicated; its regular part contains infinitely many copies of G/P. However, we can see G/P as the *Furstenberg boundary* of X, which is the set of equivalence classes of *Weyl chambers* of X for the relation "to be at finite Hausdorff distance".

Recall that a *maximal flat* of X is a flat, totally geodesic subspace of X that is maximal for inclusion. For instance, the A-orbit of the origin $x_0 :=$

$eK \in G/K = X$ is a maximal flat $\mathcal{F}_0$, and the other maximal flats are the G-translates of $\mathcal{F}_0$; they are all isometric to $\mathbb{R}^n$ where $n = \mathrm{rank}_{\mathbb{R}}(G) = \dim A$. Let $\overline{A^+}$ be the closed positive Weyl chamber of A corresponding to our choice of positive restricted root system: by definition, $\overline{A^+}$ is the subset of A on which all positive restricted roots take values ≥ 1. A *Weyl chamber* of X is a G-translate of $\overline{A^+} \cdot x_0 \subset X$. The Weyl group

$$W := N_G(A)/Z_G(A) = N_G(A)/MA$$

(where $N_G(A)$ denotes the normalizer of A in G) naturally acts on A; this induces a faithful action of W on $\mathcal{F}_0 = A \cdot x_0$, which is generated by the orthogonal reflections along some hyperplanes $\mathcal{H}_1, \ldots, \mathcal{H}_n$ of $\mathcal{F}_0$ bounding the Weyl chamber $\overline{A^+} \cdot x_0$.

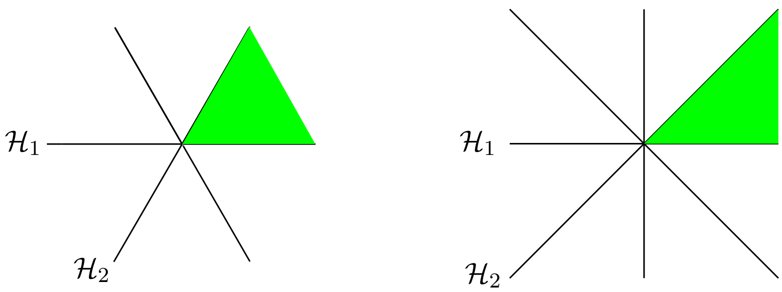


FIGURE 2. Weyl chambers in a maximal flat of X for $G = \mathrm{SL}_3(\mathbb{R})$ (type A_2) and for $G = \mathrm{Sp}_4(\mathbb{R})$ (type C_2).

As in Section 2.2, the homogeneous space G/MA identifies with the unique open G-orbit $\mathcal{O}$ in $G/P \times G/P$. (In higher rank there are more than two G-orbits.) Pairs of elements of G/P that belong to $\mathcal{O}$ are said to be *in general position.*

Example 3.2. For $G = \mathrm{SL}_{n+1}(\mathbb{R})$, the space G/P identifies with the set of maximal flags $(V_1 \subset \cdots \subset V_n)$ of $\mathbb{R}^{n+1}$, and the open G-orbit $\mathcal{O}$ is the set of pairs of maximal flags

$$(\xi, \eta) = \big((V_1 \subset \cdots \subset V_n), (V_1' \subset \cdots \subset V_n')\big)$$

such that $V_i \oplus V'_{n+1-i} = \mathbb{R}^{n+1}$ for all $1 \leq i \leq n$. For any $(\xi, \eta) \in \mathcal{O}$ we can find a basis $(e_1, \ldots, e_{n+1})$ of $\mathbb{R}^{n+1}$ that is *adapted* to (ξ, η), in the sense that

$$\begin{array}{rcl} \xi & = & \big(\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, \ldots, e_n \rangle\big), \\ \eta & = & \big(\langle e_{n+1} \rangle \subset \langle e_n, e_{n+1} \rangle \subset \cdots \subset \langle e_2, \ldots, e_{n+1} \rangle\big). \end{array}$$

• We shall identify G/P with the set of equivalence classes of Weyl chambers of X by mapping $gP \in G/P$ to $g \cdot \xi_0^+$ for all $g \in G$, where ξ_0^+ is the class of the Weyl chamber $\overline{A^+} \cdot x_0 \subset \mathcal{F}_0$. (The stabilizer of ξ_0^+ in G is P.)

• We shall identify G/MA with the open G-orbit $\mathcal{O} \subset G/P \times G/P$ by mapping $gMA \in G/MA$ to $g \cdot (\xi_0^+, \xi_0^-)$ for all $g \in G$, where ξ_0^- is the class of the Weyl chamber $(\overline{A^+})^{-1} \cdot x_0 \subset \mathcal{F}_0$. (The element ξ_0^- has stabilizer P^- in G, hence (ξ_0^+, ξ_0^-) has stabilizer $P \cap P^- = MA$.)

• The set G/MA surjects G-equivariantly onto the set of maximal flats of X by mapping (ξ_0^+,ξ_0^-) to $\mathcal{F}_0$. The fiber is the Weyl group W, acting on $\mathcal{F}_0 = A\cdot x_0$ as above.

$$\begin{array}{ccccc} G/MA & \xrightarrow{\sim} & \mathcal{O} & \overset{\text{fiber } W}{\twoheadrightarrow} & \{\text{maximal flats of } X\} \\ gMA & \longmapsto & g\cdot(\xi_0^+,\xi_0^-) & \longmapsto & g\cdot\mathcal{F}_0 \end{array} \tag{3.1}$$

We shall also consider the homogeneous space G/M, which identifies with the actual set of Weyl chambers of X (*not* modulo equivalence) with the natural action of G. When $\mathrm{rank}_{\mathbb{R}}(G)=1$, a Weyl chamber is a geodesic ray, hence is determined by its endpoint in X and its direction at the endpoint: in this case, G/M identifies with the unit tangent bundle $T^1(X)$. When $\mathrm{rank}_{\mathbb{R}}(G)\geq 2$, the space $T^1(X)$ has larger dimension than G/M:

$$\begin{aligned} \dim(T^1(X)) &= 2\dim(X)-1 \;=\; 2\dim(\mathfrak{a})+2\dim(\mathfrak{n})-1,\\ \dim(G/M) &= \dim(G)-\dim(M) \;=\; \dim(\mathfrak{a})+2\dim(\mathfrak{n}), \end{aligned}$$

where the right-hand equality in the first (resp. second) line follows from the Iwasawa decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{a}+\mathfrak{n}$ (resp. from (1.1)). As in Section 2.3, the group A acts on G/M by right multiplication. When $\mathrm{rank}_{\mathbb{R}}(G)=1$ (i.e. $\dim(A)=1$), this is the geodesic flow on $T^1(X)$. In general, the right action of A on G/M (or on $\Gamma\backslash G/M$) is called the *Weyl chamber flow.*

3.3. **Generalization of (2) and (3).** Suppose that Γ is Zariski-dense in G. By work of Benoist [Be], when $\mathrm{rank}_{\mathbb{R}}(G)\geq 2$ there still exists (as in the classical situation where $\mathrm{rank}_{\mathbb{R}}(G)=1$) a smallest, nonempty, closed, Γ-invariant subset Λ_Γ of G/P, called the *limit set* of Γ. Explicitly, Λ_Γ is the set of points $\xi\in G/P$ such that some sequence $(\gamma_n)_{n\in\mathbb{N}}$ of elements of Γ "contracts G/P towards ξ", in the sense that the push-forward by γ_n of the K-invariant probability measure ν on $G/P\simeq K/M$ converges weakly to the Dirac mass in ξ (i.e. $\nu(\gamma_n^{-1}\cdot\mathcal{U})\to 1$ for any open subset $\mathcal{U}$ of G/P containing ξ). Limit sets in higher real rank were first introduced by Guivarc'h [Gh] for $G=\mathrm{SL}_{n+1}(\mathbb{R})$.

Let A^+ be the interior of $\overline{A^+}$ in A (open positive Weyl chamber). An element $g\in G$ is said to be *proximal* if it is conjugate to an element of MA^+. In this case, it has a unique attracting fixed point ξ_g^+ and a unique repelling fixed point ξ_g^- in G/P; these two points are in general position.

Benoist proved that the limit set Λ_Γ is in fact the closure in G/P of the set of fixed points ξ_γ^+ for proximal $\gamma\in\Gamma$ [Be, Lem. 3.6.(iii)]. Moreover, the set $(\Lambda_\Gamma\times\Lambda_\Gamma)\cap G/MA$ is the closure in G/MA of the set of pairs $(\xi_\gamma^+,\xi_\gamma^-)$ for proximal $\gamma\in\Gamma$ [Be, Lem. 3.6.(iv)].

We shall consider the following generalization of Definition 2.1.(2) for convex cocompactness:

(2) Γ acts cocompactly on F_Γ,

where we define F_Γ be the union of all maximal flats of X which are images of points of $(\Lambda_\Gamma\times\Lambda_\Gamma)\cap G/MA$ under (3.1). We shall also consider the following generalization of Definition 2.1.(3) for convex cocompactness:

(3) the intersection of the $\mathfrak{a}^+$-conservative and $(-\mathfrak{a}^+)$-conservative sets of the Weyl chamber flow on $\Gamma\backslash G/M$ is compact.

Here we set $\mathfrak{a}^+ = \log A^+ \subset \mathfrak{a}$ and use the following terminology (where $\|\cdot\|$ is any fixed norm on $\mathfrak{a}$).

Definition 3.3. Let Ω be an open cone of $\mathfrak{a}$. A point $x \in \Gamma\backslash G/M$ is Ω-*conservative* if there exists a sequence $(a_n)_{n\in\mathbb{N}}$ of elements of A, tending to infinity, such that $(xa_n)_{n\in\mathbb{N}}$ is bounded in $\Gamma\backslash G/M$ and $\log(a_n)/\|\log(a_n)\|$ converges to a point of Ω. The Ω-*conservative set* of the Weyl chamber flow on $\Gamma\backslash G/M$ is the closure of the set of Ω-conservative points.

Quint established the following relations between the generalized definitions (1), (2), (3) of convex cocompactness.

Lemma 3.4 (Quint [Q]). *In any real rank,*

$$(1) \Longrightarrow (2) \Longleftrightarrow (3).$$

He also proved that in higher real rank, the definitions (2) and (3) for convex cocompactness do not give any interesting example of discrete groups apart from products of uniform lattices and convex cocompact subgroups of rank-one factors.

Theorem 3.5 (Quint [Q]). *If Γ is Zariski-dense and satisfies (2) or (3), then Γ is a product of convex cocompact subgroups of the rank-1 factors of G_1 and of a uniform lattice of G_2; in particular, any nonempty, Γ-invariant, closed, convex subset $\mathcal{C}$ of X is of the form $\mathcal{C} = \mathcal{C}_1 \times X_2$ for some Γ-invariant closed convex subset $\mathcal{C}_1$ of X_1.*

In the rest of these notes, we shall explain the essential ideas of the proof of Lemma 3.4 and Theorem 3.5.

4. Main steps in the proof of Theorem 3.5

4.1. **Proof of** (2) $\Leftrightarrow$ (3)**.** Let E'_Γ be the (full) preimage of $(\Lambda_\Gamma \cap \Lambda_\Gamma) \cap G/MA$ in G/M, and let $E_\Gamma = \Gamma\backslash E'_\Gamma \in \Gamma\backslash G/M$. In other words,

$$E_\Gamma = \left\{ \Gamma g M \in \Gamma\backslash G/M : \ g\cdot(\xi_0^+, \xi_0^-) \in \Lambda_\Gamma \times \Lambda_\Gamma \right\}. \tag{4.1}$$

By construction, E_Γ is closed and right-A-invariant. Note that the image of $E'_\Gamma \subset G/M$ in $X = G/K$ is F_Γ. Since the fiber K/M is compact, the compactness of E_Γ is equivalent to the cocompactness of the action of Γ on F_Γ.

$$\begin{tikzcd}
& E'_\Gamma & E_\Gamma \\
& \cap & \cap \\
& G/M \arrow[r] \arrow[dl] \arrow[dr] & \Gamma\backslash G/M \\
G/MA & & X = G/K \\
\cup & & \cup \\
(\Lambda_\Gamma \cap \Lambda_\Gamma) \cap G/MA & & F_\Gamma
\end{tikzcd}$$

The equivalence (2) $\Leftrightarrow$ (3) in Lemma 3.4 is an immediate consequence of this observation and of the following lemma.

Lemma 4.1 [Q, Cor. 4.2]**.** *In any real rank, E_Γ is the intersection of the $\mathfrak{a}^+$-conservative and $(-\mathfrak{a}^+)$-conservative sets of the Weyl chamber flow on $\Gamma\backslash G/M$.*

Proof. Consider $x = \Gamma g M \in \Gamma\backslash G/M$.

If x is $\mathfrak{a}^+$-conservative, then by definition there exist a sequence $(a_n) \in A^{\mathbb{N}}$ going to infinity, a bounded sequence $(\kappa_n) \in G^{\mathbb{N}}$, and a sequence $(\gamma_n) \in \Gamma^{\mathbb{N}}$, such that $\log(a_n)/\|\log(a_n)\|$ converges to a point of $\mathfrak{a}^+$ and

$$g\, a_n = \gamma_n\, \kappa_n$$

for all $n \in \mathbb{N}$. Up to passing to a subsequence, we may assume that $\kappa_n \to \kappa$ for some $\kappa \in G$. The subset $\mathcal{U}_0$ of G/P consisting of points that are in general position with ξ_0^- is open and dense in G/P. By Remark 4.2 below, for any $\xi \in G/P$ with $\kappa^{-1} \cdot \xi \in \mathcal{U}_0$,

$$\gamma_n \cdot \xi = g a_n \kappa_n^{-1} \cdot \xi \underset{n\to+\infty}{\longrightarrow} g \cdot \xi_0^+,$$

and this convergence is uniform on compact sets. Therefore, if ν is the K-invariant probability measure on G/P, we have $\nu(\gamma_n^{-1} \cdot \mathcal{U}) \to 1$ for any open subset $\mathcal{U}$ of G/P containing $g\cdot\xi_0^+$, which means by definition that $g\cdot\xi_0^+ \in \Lambda_\Gamma$. Similarly, if x is $(-\mathfrak{a}^+)$-conservative, then $g \cdot \xi_0^- \in \Lambda_\Gamma$, hence $x \in E_\Gamma$.

Conversely, suppose $x \in E_\Gamma$, i.e. $g \cdot (\xi_0^+, \xi_0^-) \in \Lambda_\Gamma \times \Lambda_\Gamma$. If $g \cdot (\xi_0^+, \xi_0^-) = (\xi_\gamma^+, \xi_\gamma^-)$ for some loxodromic element $\gamma \in \Gamma$, then $g^{-1}\gamma g \in Ma$ for some $a \in A^+$. We have $xa^n = x = xa^{-n}$ for all n, hence x is both $\mathfrak{a}^+$-conservative and $(-\mathfrak{a}^+)$-conservative. In general, we use the density [Be, Lem. 3.6.(iv)] of the set of pairs $(\xi_\gamma^+, \xi_\gamma^-)$ in $(\Lambda_\Gamma \times \Lambda_\Gamma) \cap G/MA$. □

In the proof we have used the following remark.

Remark 4.2. The subset $\mathcal{U}_0$ of G/P consisting of points that are in general position with ξ_0^- is the P^--orbit of ξ_0^+. It identifies with $P^-/(P \cap P^-)$, or equivalently with $\mathfrak{n}^-$ endowed with an action of P^- whose restriction to MA is the adjoint action ($\xi_0^+ \in \mathcal{U}_0$ corresponds to $0 \in \mathfrak{n}^-$). In particular, for any sequence $(a_n) \in A^{\mathbb{N}}$, if $\alpha(a_n) \to +\infty$ for all positive restricted roots α, then $a_n \cdot \xi \to \xi_0^+$ for all $\xi \in \mathcal{U}_0$, and the convergence is uniform on compact sets.

4.2. **Strategy of proof of Theorem 3.5.** Note that the Weyl group $W = N_G(A)/MA$ acts on G/MA by multiplication on the right.

Suppose that (2) and (3) are satisfied. Then the right-A-invariant set E_Γ is compact, hence any point of E_Γ is Ω-conservative for any open cone Ω of $\mathfrak{a}$. Taking $\Omega = w \cdot \mathfrak{a}^+$ for w ranging over W, applying Lemma 4.1, and using (4.1), we obtain that Λ_Γ is *W-stable*, in the following sense.

Definition 4.3. A subset Λ of G/P is *W-stable* if $(\Lambda \times \Lambda) \cap G/MA$ is right-W-invariant, which means that for any $g \in G$ such that $g \cdot (\xi_0^+, \xi_0^-) \in \Lambda \times \Lambda$ we have $gw \cdot \xi_0^+ \in \Lambda$ for all $w \in W$.

Theorem 3.5 therefore reduces to the following proposition.

Proposition 4.4 [Q, Prop. 3.1]**.** *The only closed, Zariski-dense, W-stable subset Λ of G/P is G/P.*

Quint first establishes Proposition 4.4 in the case that G is simple of real rank 2, i.e. for restricted root systems of type A_2 (e.g. $G = \mathrm{SL}_3(\mathbb{R})$), of type $B_2 = C_2$ (e.g. $G = \mathrm{Sp}_4(\mathbb{R})$), and of type G_2. This relies on a combinatorial argument, which we give in Section 5 for $G = \mathrm{SL}_3(\mathbb{R})$. He then reduces the general case to the rank-2 case, as we explain for $G = \mathrm{SL}_n(\mathbb{R})$ in Section 6.

5. Proof of Proposition 4.4 for $G = \mathrm{SL}_3(\mathbb{R})$

In this section we take $G = \mathrm{SL}_3(\mathbb{R})$ and see G/P as the set of pairs (p, ℓ) where p is a point of $\mathbb{RP}^2$ and $\ell \subset \mathbb{RP}^2$ a projective line containing p.

Let Λ be a closed, Zariski-dense, W-stable subset of G/P. We denote by Λ^0 (resp. Λ^1) the projection of Λ to the set of points (resp. projective lines) of $\mathbb{RP}^2$; it is Zariski-dense in $\mathbb{RP}^2$ (resp. in the space of projective lines of $\mathbb{RP}^2$) since Λ is Zariski-dense in G/P.

For any projective line $\ell \subset \mathbb{RP}^2$ and any point $q \in \mathbb{RP}^2 \smallsetminus \ell$, we call *projection from q onto ℓ* the map from $\mathbb{RP}^2 \smallsetminus \{q\}$ to ℓ that sends any projective line through q (minus $\{q\}$) to its intersection point with ℓ.

5.1. **Preliminary remarks.** Before we prove Proposition 4.4, we make a few useful observations leading to a reduction of the proposition.

(i) The W-stability of Λ means that the six flags determined by any pair of elements of Λ all belong to Λ (see Figure 3).

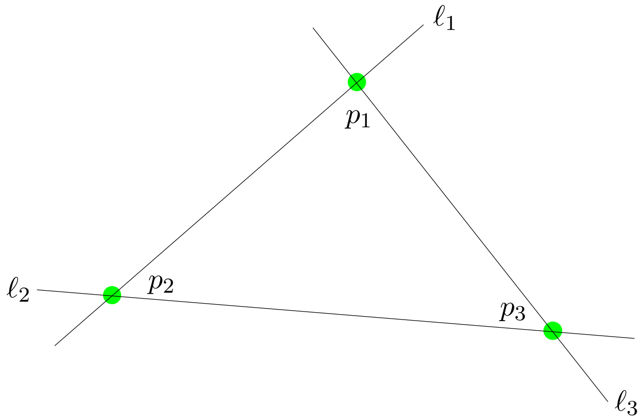


Figure 3. If (p_1, ℓ_1) and (p_2, ℓ_2) belong to the W-stable set Λ, then so do (p_1, ℓ_3), (p_2, ℓ_1), (p_3, ℓ_2), and (p_3, ℓ_3).

In particular, for any $\ell_1 \neq \ell_2$ in Λ^1 the intersection $\ell_1 \cap \ell_2$ belongs to Λ^0, for any $p_1 \neq p_2$ in Λ^0 the projective line through p_1 and p_2 belongs to Λ^1, and the projection from a point $q \in \Lambda^0$ onto a line $\ell \in \Lambda^1$ disjoint from q preserves Λ^0.

(ii) This implies that for any $\ell \in \Lambda^1$ the set $\ell \cap \Lambda^0$ is Zariski-dense in ℓ, and for any $p \in \Lambda^0$ the set of projective lines of Λ^1 through p is Zariski-dense in the set of projective lines of $\mathbb{RP}^2$ through p.

Indeed, for the first statement, fix $\ell \in \Lambda^1$ and choose an arbitrary point $q \in \Lambda^0 \smallsetminus \ell$. By (i), the projection from q onto ℓ maps the Zariski-dense subset $\Lambda^0 \smallsetminus \{q\}$ of $\mathbb{RP}^2$ to $\Lambda^0 \cap \ell$. Therefore, $\Lambda^0 \cap \ell$ is Zariski-dense in ℓ. The second statement is similar.

(iii) In order to establish Proposition 4.4 for $G = \mathrm{SL}_3(\mathbb{R})$, it is sufficient to prove the following.

Claim 5.1. *Any line $\ell \in \Lambda^1$ is contained in Λ^0, and any projective line through a point $p \in \Lambda^0$ belongs to Λ^1.*

Indeed, consider $(p, \ell) \in G/P$. Since Λ is Zariski-dense in G/P, we can find an element $(p', \ell') \in \Lambda$ in general position with (p, ℓ). Claim 5.1 gives successively that $(\ell \cap \ell') \in \Lambda^0$, that $\ell \in \Lambda^1$, and that $p \in \Lambda^0$. In other words, there exist a projective line ℓ_1 through p and a point $p_2 \in \ell$ such that (p, ℓ_1) and (p_2, ℓ) belong to Λ. Then $(p, \ell) \in \Lambda$ by (i).

5.2. **Reduction to a rank-1 result.** In order to prove Claim 5.1, we can reduce to real rank 1 and use the following lemma.

Lemma 5.2 [Q, Prop. 2.1]**.** *Suppose G has real rank 1 and let Λ be a Zariski-dense subset of G/P with the following property: for any $\xi \in \Lambda$ the group of unipotent elements $u \in G$ such that $u \cdot \xi = \xi$ and $u \cdot \Lambda = \Lambda$ acts transitively on $\Lambda \smallsetminus \{\xi\}$. Then $\Lambda = G/P$.*

Indeed, suppose that Lemma 5.2 is proved and consider $\ell \in \Lambda^1$ and $p \in \ell$. By Lemma 5.2, in order to prove that $p \in \Lambda^0$, it is sufficient to see that the set of projective transformations u of ℓ that preserve $\Lambda^0 \cap \ell$ and admit p as a unique fixed point, acts transitively on $(\Lambda^0 \cap \ell) \smallsetminus \{p\}$. (Such transformations correspond to unipotent elements of the stabilizer of ℓ, which is isomorphic to $\mathrm{SL}_2(\mathbb{R})$.) To see this, consider $p_1 \neq p_2$ in $(\Lambda^0 \cap \ell) \smallsetminus \{p\}$.
- By (ii), there exist two distinct lines $\ell_1, \ell_2 \in \Lambda^1$ through p, different from ℓ.
- By (ii) again, there exist $p_3 \neq p$ in $\ell_1 \cap \Lambda^0$. Let us denote by p_4 (resp. p_5) the image of p_1 (resp. p_2) under the projection from p_3 onto ℓ_2 (see Figure 4).
- By (i), the points p_4 and p_5 belong to Λ^0.
- By (i) again, the projection $\mathrm{proj}_{p_4,\ell_1}$ from p_4 onto ℓ_1 preserves Λ^0, and so does the projection $\mathrm{proj}_{p_5,\ell}$ from p_5 onto ℓ. In particular, the map

$$u := \mathrm{proj}_{p_5,\ell} \circ \mathrm{proj}_{p_4,\ell_1},$$

which preserves ℓ, also preserves $\Lambda^0 \cap \ell$. It maps p_1 to p_2 and admits p as a unique fixed point. This completes the proof of Claim 5.1, hence of Proposition 4.4 for $G = \mathrm{SL}_3(\mathbb{R})$.

5.3. **Proof of Lemma 5.2.** We may assume that G is connected. Let S be the stabilizer of Λ in G; it is a closed subgroup of G.
- S is Zariski-dense in G. Indeed, let H be the identity component (for the real topology) of the Zariski closure of G. The flag variety of H is a Zariski-closed subset of G/P; by assumption, it contains the Zariski-dense set Λ, hence it is equal to G/P. This implies $H = G$.
- S is not discrete in G. Indeed, since Λ is infinite and G/P is compact, Λ admits a nonisolated point η: there exists a sequence $(\eta_n)_{n\in\mathbb{N}}$ of pairwise distinct points of Λ converging to η. By assumption, for any n there exists a unipotent element $u_n \in S$ such that $u_n \cdot \xi = \xi$ and $u_n \cdot \eta = \eta_n$. The sequence $(u_n)_{n\in\mathbb{N}}$ converges to 1, hence S is not discrete.

This implies that $S = G$ (a Zariski-dense subgroup of a connected semisimple Lie group is either discrete or dense), and so $\Lambda = G/P$.

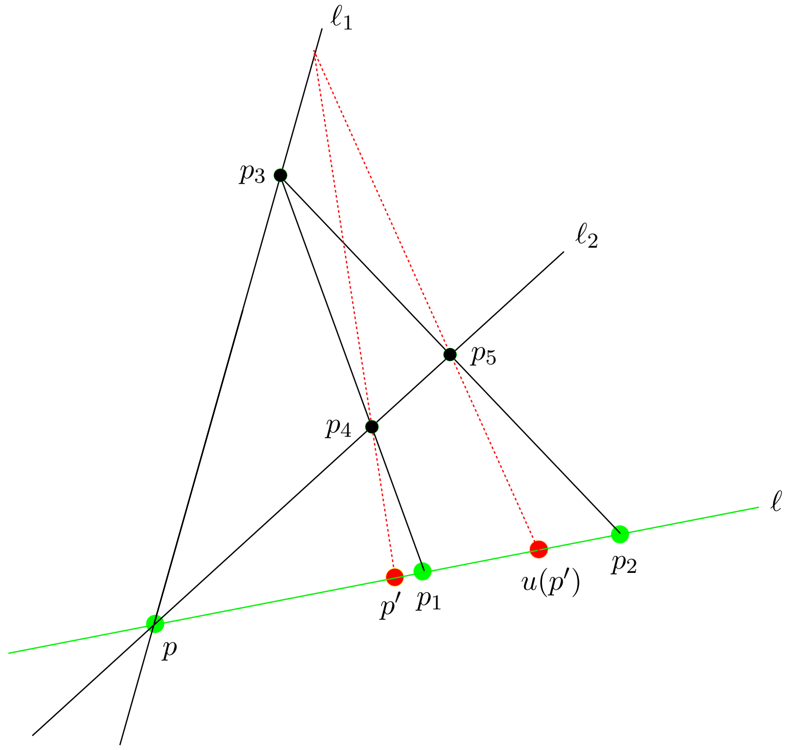


Figure 4. Construction of a unipotent transformation u.

6. Reducing to rank-2 groups in Proposition 4.4

To simplify notation and make the proof more concrete, we suppose that $G = \mathrm{SL}_{n+1}(\mathbb{R})$, that A is the subgroup of diagonal matrices with positive entries, and that P is the group of upper triangular matrices (Example 1.1). The space G/P identifies with the set of maximal flags

$$(V_1 \subset \cdots \subset V_n)$$

of $\mathbb{R}^{n+1}$ (Example 3.2). Let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, for $1 \leq i \leq n$, be the standard simple roots of A in G. For any i, let P_i be the parabolic subgroup of G associated with $\{\alpha_i\}$, consisting of upper triangular block matrices with square diagonal blocks of size $1, \ldots, 1, 2, 1, \ldots, 1$, where 2 is the i-th entry. For $\xi = g \cdot \xi_0^+ \in G/P$, we set

$$P_i^\xi = gP_ig^{-1}.$$

Let Λ be a closed, Zariski-dense, W-stable subset of G/P. To show that $\Lambda = G/P$, we first note that for any $\xi, \eta \in G/P$ in general position, there is a finite sequence

$$\xi = \xi_1,\ \xi_2,\ \ldots,\ \xi_{k+1} = \eta$$

of elements of G/P such that for any $1 \leq j \leq k$ we have $\xi_{j+1} \in P_{i_j}^{\xi_j} \cdot \xi_j$ for some $1 \leq i_j \leq n$. Indeed, write the longest element

$$w_0 : (1, \ldots, n+1) \longmapsto (n+1, \ldots, 1)$$

of the Weyl group $W \simeq \mathfrak{S}_{n+1}$ as a product $\tau_1 \ldots \tau_k$ of transpositions $\tau_j : (i_j, i_j+1) \mapsto (i_j+1, i_j)$; if $(\xi, \eta) = g \cdot (\xi_0^+, \xi_0^-)$, then we may take

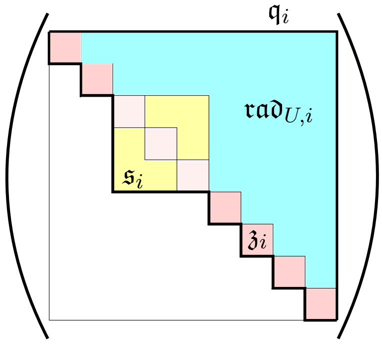
$$\xi_{j+1} := g\tau_1 \ldots \tau_j \cdot \xi_0^+$$

for all j. By induction, it is sufficient to prove that if $\xi_j \in \Lambda$, then $\xi_{j+1} \in \Lambda$.

Let us prove a stronger statement, namely that for any $\xi \in \Lambda$ and any $1 \leq i \leq n-1$ we have $Q_i^\xi \cdot \xi \subset \Lambda$, where Q_i^ξ is the parabolic subgroup of G generated by P_i^ξ and P_{i+1}^ξ. To simplify notation, we assume that $\xi = \xi_0^+$ (the proof for general ξ is similar). The group $Q_i := Q_i^{\xi_0^+}$ admits a Levi decomposition

$$Q_i = Z_i\, S_i \ltimes \mathrm{Rad}_{U,i},$$

where $\mathrm{Rad}_{U,i} \subset N$ is the unipotent radical of Q_i (i.e. the largest connected normal unipotent subgroup), $Z_i \subset A$ is the center of Q_i (consisting of diagonal matrices whose i-th, $(i+1)$-th, and $(i+2)$-th entries are equal), and S_i is a semisimple Lie group (isomorphic to $\mathrm{SL}_3(\mathbb{R})$). The orbit $Q_i \cdot \xi_0^+$ identifies with $Q_i/(P \cap Q_i)$ and, seen as an S_i-homogeneous space, with $S_i/(P \cap S_i)$, which is the flag variety of $S_i \simeq \mathrm{SL}_3(\mathbb{R})$.



In order to prove that $Q_i \cdot \xi_0^+ \subset \Lambda$, assuming $\xi_0^+ \in \Lambda$, it is enough (by the rank-2 case of Proposition 4.4 treated in Section 5) to prove the following.

Claim 6.1. *The set $\Lambda \cap (Q_i \cdot \xi_0^+)$, seen as a subset of the flag variety of S_i, is closed, Zariski-dense, and stable under the Weyl group of S_i.*

Proof. Let $\mathrm{pr}_1 : G/MA \simeq \mathcal{O} \subset G/P \times G/P \to G/P$ be the projection onto the first factor and let $w_i \in W \simeq \mathfrak{S}_{n+1}$ be the transposition $(i, i+2) \mapsto (i+2, i)$. Recall that the Weyl group $W = N_G(A)/MA$ acts on G/MA on the right. For any $\xi \in G/P$, let $\mathcal{U}_\xi$ be the (Zariski-dense) set of elements of G/P that are in general position with ξ. We define a map $\pi_i^\xi : \mathcal{U}_\xi \to G/P$ by

$$\pi_i^\xi(\eta) = \mathrm{pr}_1\big((\xi, \eta) \cdot w_i\big) \tag{6.1}$$

for all $\eta \in \mathcal{U}_\xi$. Concretely, if $(e_1, \ldots, e_{n+1})$ is a basis of $\mathbb{R}^{n+1}$ adapted to (ξ, η) (see Example 3.2), then

$$\pi_i^\xi(\eta) = \big(V_1 \subset \cdots \subset V_{i-1} \subset V_i' \subset V_{i+1}' \subset V_{i+2} \subset \cdots \subset V_n\big) \tag{6.2}$$

where $\xi = (V_1 \subset \cdots \subset V_n)$ and

$$V_i' := V_{i-1} + \mathbb{R}\, e_{i+2}, \qquad V_{i+1}' := V_{i-1} + \mathbb{R}\, e_{i+1} + \mathbb{R}\, e_{i+2}. \tag{6.3}$$

We now restrict to $\xi \in Q_i \cdot \xi_0^+$. From (6.2) and (6.3) we see that the set $\pi_i^\xi(\mathcal{U}_\xi)$ is contained in $Q_i^\xi \cdot \xi = Q_i \cdot \xi_0^+$, and in fact identifies with the set of elements of $S_i/(P \cap S_i)$ that are in general position with the image of ξ; in particular, $\pi_i^\xi(\mathcal{U}_\xi)$ is Zariski-dense in $Q_i \cdot \xi_0^+ \simeq S_i/(P \cap S_i)$. From (6.1) we see that:

- π_i^ξ is an algebraic homomorphism; in particular, since $\Lambda \cap \mathcal{U}_\xi$ is Zariski-dense in $\mathcal{U}_\xi$, the set $\pi_i^\xi(\Lambda \cap \mathcal{U}_\xi)$ is Zariski-dense in $\pi_i^\xi(\mathcal{U}_\xi)$, hence in $Q_i \cdot \xi_0^+$;
- if $\xi \in \Lambda$, then $\pi_i^\xi(\Lambda \cap \mathcal{U}_\xi) \subset \Lambda$ since Λ is W-stable.

Therefore, the set $\Lambda \cap (Q_i \cdot \xi_0^+)$ is Zariski-dense in $Q_i \cdot \xi_0^+$. Let us prove that it is W_i-stable as a subset of $S_i/(P \cap S_i)$, where

$$W_i = N_{S_i}(A \cap S_i)/Z_{S_i}(A \cap S_i)$$

is the Weyl group of S_i, which embeds in W as the subgroup generated by the transpositions $(i, i+1) \mapsto (i+1, i)$ and $(i+1, i+2) \mapsto (i+2, i+1)$. From (6.1) and the W-stability of Λ we see that for any $\xi \in \Lambda \cap (Q_i \cdot \xi_0^+)$ and any $\eta \in \Lambda \cap \mathcal{U}_\xi$ we have $(\xi, \pi_i^\xi(\eta)) \cdot w_i' \in \Lambda \times \Lambda$ for all $w_i' \in W_i$. Therefore, it is sufficient to prove that for any pair (ξ, ζ) of points of $\Lambda \cap (Q_i \cdot \xi_0^+)$ that are in general position in $Q_i \cdot \xi_0^+ \simeq S_i/(P \cap S_i)$, we can write $\zeta = \pi_i^\xi(\eta)$ for some $\eta \in \Lambda \cap \mathcal{U}_\xi$. Fix such a pair (ξ, ζ) and consider a point $\tau \in \Lambda \cap \mathcal{U}_\xi \cap \mathcal{U}_\zeta$; then we may take $\eta := \pi_{n-i}^\tau(\zeta)$. $\square$

7. Proof of (1) ⇒ (2) in Lemma 3.4

In order to prove that (1) implies (2), it is sufficient to check the following.

Lemma 7.1 [Q, Lem. 5.4]**.** *The set F_Γ is contained in any nonempty, Γ-invariant, closed, convex subset $\mathcal{C}$ of $X = G/K$.*

Recall from Section 3.3 that F_Γ is the set of maximal flats of X corresponding to elements of $(\Lambda_\Gamma \times \Lambda_\Gamma) \cap G/MA$ under (3.1). For $(\xi, \eta) \in G/MA$, we shall denote by $\mathcal{F}(\xi, \eta)$ the corresponding maximal flat of X.

Recall that any element of G admits a *Jordan decomposition*: $g = g_h g_e g_u$ for some unique commuting elements $g_h, g_e, g_u \in G$ with g_h hyperbolic (i.e. conjugate to an element $a_g \in \overline{A^+}$), with g_e elliptic (i.e. conjugate to an element of M), and with g_u unipotent. We denote by $\lambda : G \to \overline{\mathfrak{a}^+} := \log \overline{A^+}$ the *Lyapunov projection* of G, defined by $\lambda(g) = \log a_g$ for all g. By definition, the *limit cone* $\mathcal{L}_\Gamma$ is the smallest closed cone of $\overline{\mathfrak{a}^+}$ containing $\lambda(\Gamma)$. Benoist [Be] proved that if Γ is Zariski-dense in G, then $\mathcal{L}_\Gamma$ is convex with nonempty interior.

Sketch of proof of Lemma 7.1. The set of pairs $(\xi_\gamma^+, \xi_\gamma^-)$ for loxodromic $\gamma \in \Gamma$ is dense in $(\Lambda_\Gamma \times \Lambda_\Gamma) \cap G/MA$ [Be, Lem. 3.6.(iv)]. Therefore, by convexity of $\mathcal{C}$, it is sufficient to prove that $\mathcal{F}(\xi_\gamma^+, \xi_\gamma^-) \subset \mathcal{C}$ for any loxodromic $\gamma \in \Gamma$. This is done in four steps:

(a) The flat $\mathcal{F}(\xi_\gamma^+, \xi_\gamma^-)$ meets $\mathcal{C}$. Indeed, for any point $x \in \mathcal{C}$, the geodesic segments $[\gamma^n \cdot x, \gamma^{-n} \cdot x]$, for $n \in \mathbb{N}$, are contained in $\mathcal{C}$ and converge (up to passing to a subsequence) to a geodesic line in $\mathcal{F}(\xi_\gamma^+, \xi_\gamma^-) \cap \mathcal{C}$ (see [Q, Lem. 5.2]).

(b) By [Be, Lem. 4.2 bis], for any $v \in \mathcal{L}_\Gamma$ with $\|v\| = 1$ there is a sequence $(\gamma_n)_{n\in\mathbb{N}}$ of loxodromic elements of Γ such that $\lambda(\gamma_n)/\|\lambda(\gamma_n)\| \to v$ and $(\xi^+_{\gamma_n}, \xi^-_{\gamma_n}) \to (\xi^+_\gamma, \xi^-_\gamma)$.

(c) This implies that for any $v \in \mathcal{L}_\Gamma$ with $\|v\| = 1$, the set $\mathcal{F}(\xi^+_\gamma, \xi^-_\gamma) \cap \mathcal{C}$ contains a geodesic of direction v (after identification of $\mathcal{F}(\xi^+_\gamma, \xi^-_\gamma)$ with $\mathcal{F}_0$, hence with $\mathfrak{a}$, which is well-defined up to the action of the Weyl group W). Indeed, fix $x \in \mathcal{F}(\xi^+_\gamma, \xi^-_\gamma) \cap \mathcal{C}$. For any $n \in \mathbb{N}$, there is a geodesic of direction $\lambda(\gamma_n)$ in $\mathcal{F}(\xi^+_{\gamma_n}, \xi^-_{\gamma_n}) \cap \mathcal{C}$ whose distance to x is bounded independently of n (see [Q, Lem. 5.2]). Up to passing to a subsequence, these lines converge to a geodesic of direction v in $\mathcal{F}(\xi^+_\gamma, \xi^-_\gamma) \cap \mathcal{C}$.

(d) The limit cone $\mathcal{L}_\Gamma$ has nonempty interior [Be, Th. 1.2]. By (c), the set $\mathcal{F}(\xi^+_\gamma, \xi^-_\gamma) \cap \mathcal{C}$ contains lines whose directions form a cone with nonempty interior. This implies $\mathcal{F}(\xi^+_\gamma, \xi^-_\gamma) \subset \mathcal{C}$ by convexity of $\mathcal{C}$. □

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LECTURE NOTES FOR CONVEX REAL PROJECTIVE STRUCTURES

QIONGLING LI

Introduction

This short note consists of two parts.

Part I is to introduce the convex $\mathbb{R}P^2$-structures on a compact surface and to explain that the deformation space of such structures on a closed surface S of genus $g > 1$ form a moduli space $\mathcal{B}(S)$ homeomorphic to an open cell of dimension $16(g-1)$. This work is done by Choi and Goldman.

Part II is to explain the correspondence between the deformation space $\mathcal{B}(S)$ and the space of pairs (Σ, U), where Σ is a Riemann surface varying in Teichmüller space and U is a cubic differential on Σ. Teichmüller space $\mathcal{T}(S)$ embeds inside $\mathcal{B}(S)$ as the locus of pairs $(\Sigma, 0)$, where Σ is a Riemann surface varying in Teichmüller space. This work is independently done by Labourie and Loftin.

Part I: Deformation space of convex $\mathbb{R}P^2$-structures

1. Convex $\mathbb{R}P^2$-structures

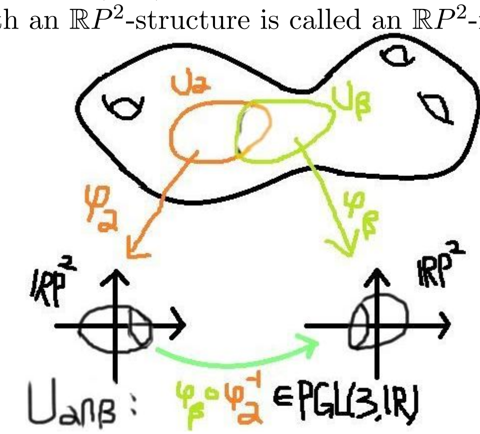
Definition 1. An $\mathbb{R}P^2$-structure on a smooth 2-manifold M is a system of coordinate charts $\{(U_\alpha, \phi_\alpha)\}$ satisfying

(i) $\{U_\alpha\}$ is an open cover of M;

(ii) $\psi_\alpha : U_\alpha \to \mathbb{R}P^2$ is a diffeomorphism onto its image;

(iii) $\psi_\beta \circ {\psi_\alpha}^{-1} \in PGL(3, \mathbb{R})$.

A manifold with an $\mathbb{R}P^2$-structure is called an $\mathbb{R}P^2$-maniflod.



Definition 2. An $\mathbb{R}P^2$-structure on M is called convex if its developing map is a diffeomorphism of $\widetilde{M}$ onto a convex domain Ω in some affine $\mathbb{R}^2 \subset \mathbb{R}P^2$. In this case, we can realize $M = \Omega/\Gamma$, where Γ is a subgroup of $PGL(3, \mathbb{R})$ which acts discretely and properly discontinuous on Ω.

Fact. *Let $M = \Omega/\Gamma$ be a closed surface with a convex $\mathbb{R}P^2$-structure. Suppose that $\chi(M) < 0$. Then the following hold.*
(1) $\Omega \subset \mathbb{R}P^2$ is a strictly convex domain with C^1 boundary and therefore contains no affine line.
(2) Either $\partial\Omega$ is a conic in $\mathbb{R}P^2$ or in not $C^{1+\epsilon}$ for some $0 < \epsilon < 1$.
(3) The attracting and repelling fixed points of elements of Γ form a dense subset of $\partial\Omega$. Furthermore given any pair $(x, y) \in \partial\Omega \times \partial\Omega$, there exists a sequence $\gamma_n \in \Gamma$ such that $Fix_+(\gamma_n) \to x$, and $Fix_-(\gamma_n) \to y$.

Let S denote a compact surface.

Definition 3. $\mathbb{R}P^2(S) := \{(f, M)|f : S \to M$ is a diffeomorphism and M is an $\mathbb{R}P^2$-manifold$\}/$, where (f, M) (f', M') if there exists a projective isomorphism $h : M \to M'$ such that $h \circ f$ is isotopic to f'.

$\mathcal{B}(S)$ denotes the subset of $\mathbb{R}P^2(S)$ corresponding to convex $\mathbb{R}P^2$-structures.

Fact. *(1) The set of equivalent classes $\mathbb{R}P^2(S)$ have a natural topology making it locally equivalent to $Hom(\pi, PGL(3, \mathbb{R}))/PGL(3, \mathbb{R})$, i.e., there exists a holonomy map $hol : \mathbb{R}P^2(S) \to Hom(\pi, PGL(3, \mathbb{R}))/PGL(3, \mathbb{R})$ which is a local diffeomorphism.*
(2) $\mathcal{B}(S)$ is open in $\mathbb{R}P^2(S)$.

Excercise 1. *Show the following.*
(1) $\mathbb{R}P^2(S)$ is a Hausdorff real analytic manifold of dimension $-8\chi(S)$;
(2) The restriction of $hol : \mathbb{R}P^2(S) \to Hom(\pi, SL(3, \mathbb{R}))/SL(3, \mathbb{R})$ to $\mathcal{B}(S)$ is an embedding of $\mathcal{B}(S)$ onto a Hausdorff real analytic manifold of dimension $-8\chi(S)$.

Our main goal to show the following theorem.

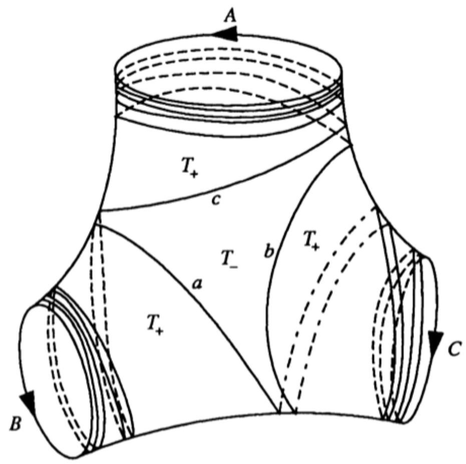
MainTheorem. *(Goldman [3]) Let S be a compact surface having a boundary components such that $\chi(S) < 0$. Then $\mathcal{B}(S)$ is diffeomorphic to a celll of dimension $-8\chi(S)$ and the map which associates to a convex $\mathbb{R}P^2$-structures neaat ∂M is a fibration of $\mathcal{B}(S)$ over an open $2n$-cell with fiber an open cell of dimension $-8\chi(S) - 2n$.*

Corollary 1.1. *Let S be a closed orientable surface of genus $g > 1$. Then $\mathcal{B}(S)$ is diffeomorphic to an open celll of dimension $16(g - 1)$.*

To prove the above main theorem, we cut a surface into pair-of-pants as what we did in Teichmueller theory. We first try to understand the simplest case when where S is a pair-of-pants. Then if we know how to assemble the convex $\mathbb{R}P^2$-structures on pair-of-pants together, especially if the convex property is preserved (which is indeed true), we are almost done.

2. Convex $\mathbb{R}P^2$-structures on a pair-of-pants

In this section, we prove Theorem 1 in the case where S is a pair-of-pants with A, B, C three components.

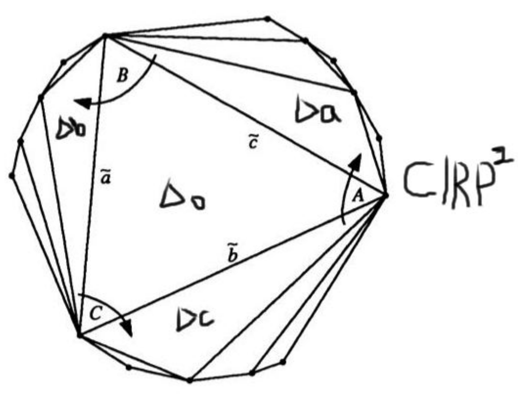


Theorem 2.1. *The deformation space $\mathcal{B}(S)$ of convex $\mathbb{R}P^2$-structures on S is an open 8-dimension cell.*

Sketch of Proof.

Suppose that $A \in SL(3, \mathbb{R})$. We define $\lambda(A)$ to be the real eigenvalue of $A \in SL(3, \mathbb{R})$ having the smallest absolute value and $\tau(A) \in \mathbb{R}$ as the sum of the other two (possibly unreal) eigenvalues. We can show that the pair $(\lambda(A), \tau(A))$ is a complete invariant of the $SL(3, \mathbb{R})$-conjugacy class of the boundary A. Such pairs form a 2-cell.

Denote $\mathcal{O}$ as the space of $SL(3, \mathbb{R})$-conjugation classes of the set of all $(\triangle_0, \triangle_a, \triangle_b, \triangle_c, A, B, C)$ satisfying conditions described in following picture.



$CBA = I$

the union $\overline{\Delta}_0 \cup \overline{\Delta}_a \cup \overline{\Delta}_b \cup \overline{\Delta}_c$ is a convex hexagon;

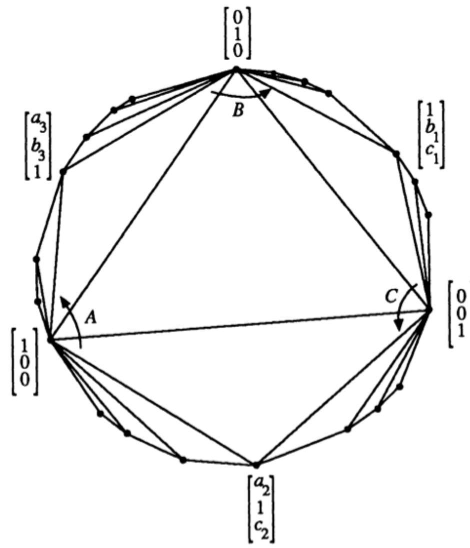
The reason we consider $\mathcal{O}$ instead of $\mathcal{B}(S)$ is because:

(1) $\mathcal{O}$ is computable and,

(2) Each $\mathbb{R}P^2$-structure corresponding to a point in $\mathcal{O}$ is convex, i.e.,$\mathcal{O} = \mathcal{B}(S)$. (the claim needs more proof.)

Now it is sufficient to show the map $\theta_{\partial S} : \mathcal{B}(S) \to \mathcal{B}(\partial S)$ obtained by associating to a convex structure the boundary invariants $((\lambda, \tau)_A, (\lambda, \tau)_B, (\lambda, \tau)_C)$ is a fibration over an open 6-cell with fiber a 2-dimensional open cell.

Since $\mathcal{O}$ considers the conjugation class under $SL(3, \mathbb{R})$, then we can consider the set $(\triangle_0, \triangle_a, \triangle_b, \triangle_c)$ as following picture parametrized by $(b_1, c_1, a_2, b_2, a_3, b_3)$. Note this set is not fixed under diagonal matrices $\text{diag}(\lambda, \mu, \nu)$ with $\lambda\mu\nu = 1$. So actually we have 4 dimensional parameters for the conjugation clases $(\triangle_0, \triangle_a, \triangle_b, \triangle_c)$.



Here A is parametrized by new parameters $(\alpha_1, \beta_1, \gamma_1)$, written as

$$A = \alpha_1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot [1 \quad a_2 \quad 0] + \beta_1 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot [0 \quad -1 \quad 0]$$

$$+ \gamma_1 \cdot \begin{bmatrix} a_3 \\ b_3 \\ -1 \end{bmatrix} \cdot [0 \quad c_2 \quad 1]$$

$$= \begin{bmatrix} \alpha_1 & \alpha_1 a_2 + \gamma_1 a_3 c_1 & \gamma_1 a_3 \\ 0 & -\beta_1 + \gamma_1 b_3 c_2 & \gamma_1 b_3 \\ 0 & -\gamma_1 c_2 & -\gamma_1 \end{bmatrix},$$

Here B is parametrized by new parameters $(\alpha_2, \beta_2, \gamma_2)$, written as

$$B = \alpha_2 \cdot \begin{bmatrix} -1 \\ b_1 \\ c_1 \end{bmatrix} \cdot [1 \quad 0 \quad a_3] + \beta_2 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot [0 \quad 1 \quad b_3]$$
$$+ \gamma_2 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot [0 \quad 0 \quad -1]$$
$$= \begin{bmatrix} -\alpha_2 & 0 & -\alpha_2 a_3 \\ \alpha_2 b_1 & \beta_2 & \beta_2 b_3 + \alpha_2 a_3 b_1 \\ \alpha_2 c_1 & 0 & -\gamma_2 + \alpha_2 a_3 c_1 \end{bmatrix},$$

Here C is parametrized by new parameters $(\alpha_3, \beta_3, \gamma_3)$, written as

$$C = \alpha_3 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot [-1 \quad 0 \quad 0] + \beta_3 \cdot \begin{bmatrix} a_2 \\ -1 \\ c_2 \end{bmatrix} \cdot [b_1 \quad 1 \quad 0]$$
$$+ \gamma_3 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot [c_1 \quad 0 \quad 1]$$
$$= \begin{bmatrix} -\alpha_3 + \beta_3 a_2 b_1 & \beta_3 a_2 & 0 \\ -\beta_3 b_1 & -\beta_3 & 0 \\ \gamma_3 c_1 + \beta_3 b_1 c_2 & \beta_3 c_2 & \gamma_3 \end{bmatrix},$$

From $\det(A) = \det(B) = \det(C) = 1$, we obtain that

$$\alpha_1 \beta_1 \gamma_1 = \alpha_2 \beta_2 \gamma_2 = \alpha_3 \beta_3 \gamma_3 = 1$$

From $CBA = 1$, we obtain that

$$\alpha_1 \alpha_2 \alpha_3 = \beta_1 \beta_2 \beta_3 = \gamma_1 \gamma_2 \gamma_3 = 1.$$

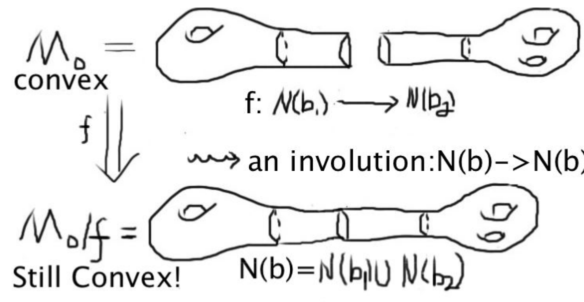
The above six equations actually can be reduced to 5 restrictions. Hence considering (A, B, C) gives us 9-5=4 more parameters. Hence we can see $\dim(\mathcal{O}) = 4 + 4 = 8$.

We are not finished since we need to know $\mathcal{O}$ is a cell. Actually, if we are careful, we may be able to choose two more right parameters (s, t) valued in $\mathbb{R}_+$ besides the boundary invariants $((\lambda, \tau)_A, (\lambda, \tau)_B, (\lambda, \tau)_C)$, then we are done.

3. Assembling convex $\mathbb{R}P^2$-structures

The proof is based on the Fenchel-Nilsen coordinate system on Teichmueller space. The proof of Main Theorem is based on the following two lemmas,

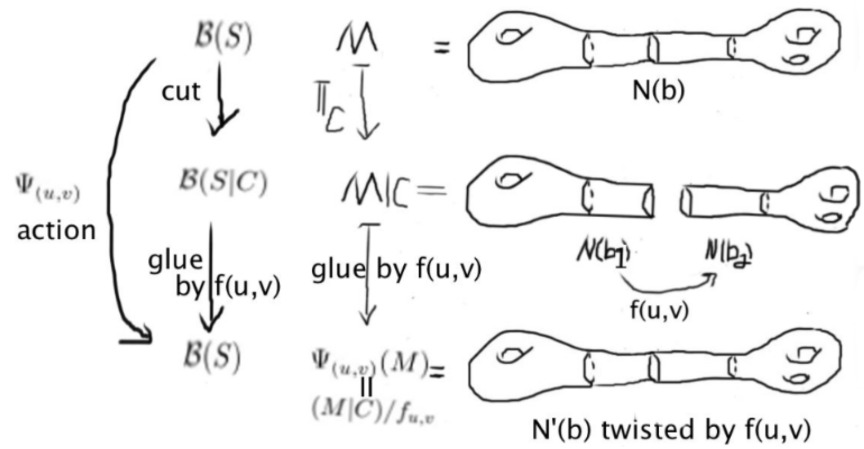
Lemma 3.1. (***Gluing Convex*** $\mathbb{R}P^2$***-manifolds***) *Let M_0 be a compact convex $\mathbb{R}P^2$-manifold with principal boundary, and suppose that $b_1, b_2 \subset \partial M_0$ are boundary components with collar neighborhoods $b_i \subset N(b_i) \subset M_0 (i = 1, 2)$. Suppose that $f : N(b_1) \to N(b_2)$ is a projective isomorphism. Then the $\mathbb{R}P^2$-manifold M_0/f obtained by identification by $f : N(b_1) \to N(b_2)$ is a **convex** $\mathbb{R}P^2$-manifold.*



From the above lemma, we know that gluing convex $\mathbb{R}P^2$-structures along a geodesic curve C only depends on the projective isomorphism f on the neighborhood of C. Then conversely we actually know how much information we lose if we cut a convex $\mathbb{R}P^2$-structure along a geodesic curve, which is stated in the following lemma.

Lemma 3.2. (***Gut Convex*** $\mathbb{R}P^2$***-manifolds***) *Suppose that* $C \subset S$ *is a two-sided simple closed curve such that each component of* $S|C$ *has negative Euler characteristic. Let* $\Pi_C : \mathcal{B}(S) \to$ *a subspace of* $\mathcal{B}(S|C)$ *be the map which arises from splitting a convex* $\mathbb{R}P^2$*-structure on* S *along the closed geodesic homotopic to* C*. Then* Π_C *is a fibration and each fiber of is an open 2-cell.*

Sketch of Lemma: Let $b_1, b_2 \subset \partial(S|C)$ be the two boundary components corresponding to C. We now define an $\mathbb{R}^2$-action Ψ on $\mathcal{B}(S)$ such that Π_C is a Ψ-invariant fibration onto the subspace of $\mathcal{B}(S|C)$ defined by the conditions $(\lambda, \tau)_{b_1} = (\lambda, \tau)_{b_2}$ and Ψ is simply transitive on each fiber of Π_C. For $(u, v) \in \mathbb{R}^2$ we construct a new convex $\mathbb{R}P^2$-manifold $\Psi_{(u,v)}(M)$ representing a point in $\mathbb{R}P^2$-manifold.
Consider the split $\mathbb{R}P^2$-manifold $M|C$; let $b_1, b_2 \subset \partial(M|C)$ be the two boundary components corresponding to C. For any $(u, v) \in \mathbb{R}^2$, there exists a principal collar neighborhoods $N(b_i) \subset M|C$ of b_i for $i = 1, 2$



Let $T^u = \begin{pmatrix} e^{-u} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^u \end{pmatrix}, U^v = \begin{pmatrix} e^{-v} & 0 & 0 \\ 0 & e^{2v} & 0 \\ 0 & 0 & e^{-v} \end{pmatrix}$, where $u, v \in \mathbb{R}$. Let $f(u,v) : N(b_1) \to N(b_2)$ be a projective isomorphism such that $f(u,v)$ is induced by $T^u U^v$ on the developing image in $\mathbb{R}P^2$. As in Gluing Lemma, there is a corresponding convex $\mathbb{R}P^2$-manifold $(M|C)/f(u,v)$ representing a point $\Psi_{(u,v)}(M)$ in $\mathcal{B}(S)$. This action generalizes the Fenchel-Nilsen twist flows on the Teichmueller space.

MainTheorem. *(Goldman [3]) Let S be a compact surface having a boundary components such that $\chi(S) < 0$. Then $\mathcal{B}(S)$ is diffeomorphic to a celll of dimension $-8\chi(S)$ and the map which associates to a convex $\mathbb{R}P^2$-structures neaat ∂M is a fibration of $\mathcal{B}(S)$ over an open $2n$-cell with fiber an open cell of dimension $-8\chi(S) - 2n$.*

4. Hitchin Component

Hitchin shows that $Hom(\pi, SL(3,\mathbb{R}))/SL(3,\mathbb{R})$ has exactly three connected components:
1. $\mathcal{C}_0$, the component containing the class of the trivial representation;
2. $\mathcal{C}_1$, the component consisting of classes of representatons which do not lift to the double covering of $PGL(3,\mathbb{R})$;
3. $\mathcal{C}_2$, the component containing faithful representations into $SO(2,1)$.

In particaular, the component $\mathcal{C}_2$ contais the Teichmueller space of S. From our exercise, we know the holonomy map **hol** maps $\mathcal{B}(S)$ bijectively onto an open subset in $\mathcal{C}_2$. Furthermore, Hitchin shows that $\mathcal{C}_2$ is homeomorphic to $\mathbb{R}^{16(g-1)}$. This naturally let Hitchin to conjecture that $\mathcal{C}_2 = \mathcal{B}(S)$.

Theorem 4.1. *(Choi and Goldman [2]) $\mathcal{B}(S)$ is closed in $\mathcal{C}_2$. Hence $\mathcal{C}_2 = \mathcal{B}(S)$.*

Part II: Correspondence of $\mathcal{B}(S)$ and spaces of pairs (Σ, U)

From now on, let S and M be a closed orientable surface of genus g. Our goal is to show the following theorem.

MainTheorem. *(Loftin [5], Labourie [4]) There exists a natural bijective correspondence betwen convex $\mathbb{R}P^2$-structures on S and pairs (Σ, U), where Σ is a Riemann surface homeomorphic to S, and U is a holomorphic cubic differential on Σ.*

Benoist and Hulin in [1] generalize the above result to the case when S is not necessarily closed stated as follows.

Theorem 4.2. *(Theorem 1.1 in [1]) Let S be an oriented surface with non-abelian fundamental group. The map $\mathcal{A} \to (J, U)$ is a bijection between:*
1. the set of properly convex projective structures $\mathcal{A}$ on s with finite Finster volume;
2. the set of hyperbolic Riemann surface structures J on S with finite volume

together with a holomorphic cubic differential U on (S, J) with poles of order at most 2 at the cusps.

5. Hyperbolic Affine Sphere

This section is mainly copied from Loftin [5].
Consider a hypersurface immersion $f : H \to \mathbb{R}^3$, and consider a transversal vector field ξ on the hypersurface H. We have the equations:

$$\begin{aligned} D_X f_*(Y) &= f_*(\nabla_X Y) + h(X,Y)\xi \\ D_X \xi &= -f_*(SX) + \beta(X)\xi. \end{aligned}$$

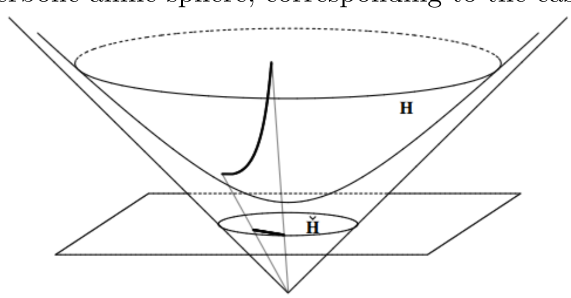
Here, X and Y are tangent vectors on H, the operator D is the canonical flat connection induced from $\mathbb{R}^3$, the operator ∇ is a torsion-free connection, the form h is a symmetric bilinear form on $T_x(H)$, the map S is an endormorphism of $T_x(M)$, and β is a one-form.

For the case the hypersurface H is strictly convex and ξ is the affine normal (see definiton in page 8 in [5]), the structure equations for H become

$$\begin{aligned} D_X f_*(Y) &= f_*(\nabla_X Y) + g(X,Y)\xi \\ D_X \xi &= -f_*(SX). \end{aligned}$$

The connection ∇ is callled the Blaschke connection. The bilinear form h is called the affine metric (since H is strictly convex)and the endormorphism S is called the affine shape operator.

Definition 4. An affine sphere is a hypersurface H in $\mathbb{R}^3$ all of whose affine normals point toward a given point in $\mathbb{R}P^2$, the center of the affine sphere, in this case $S = LI$, where the affine mean curvature L is a constant function on H and I is the identity map. If the center lies on the concave side of H, it is called a hyperbolic affine sphere, corresponding to the case L is negative.



Thus by scaling, we can normalize any hyperbolic affine sphere to have $L = -1$. Also, we can translate so that the center is 0. Then the affine normal $\xi = f$, where f is the embedding of H into $\mathbb{R}^3$. The structure equations then become

$$\begin{aligned} D_X f_*(Y) &= f_*(\nabla_X Y) + g(X,Y) f \\ D_X f &= f_*(X). \end{aligned}$$

Proposition 5.1. *Consider a convex, bounded domain $\Omega \subset \mathbb{R}^2$, where $\mathbb{R}^2$ is embedded in $\mathbb{R}^3$ as the affine space $x_3 = 1$. Then, there is a unique properly embedded hyperbolic affne sphere $H \subset \mathbb{R}^3$ of affine mean curvatuer -1 and center 0 asymptotic to the boundary of the cone $\mathcal{C}(\Omega) \subset \mathbb{R}^3$.*

Proposition 5.2. *Each convex $\mathbb{R}P^2$-structure on M corresponds to an affine sphere structure on M(i.e., M can be realized as a quotient of a byperbolic affine sphere with center 0 and affine mean curvature -1 in the cone $\mathcal{C}$.) vice versa.*

Now we make use of the affine sphere structures to get the correspondence.

6. How to get a holomorphic cubic differential

This section is mainly copied from Loftin [6].
To get a holomorphic cubic differential from this construction, we begin to work with complexified tangent vectors, and we extend ∇, h and D by complex linearity. Consider a local conformal coordinate $z = x + iy$ on the hyperbolic affine sphere with center the origin and affine curvature -1. Then the affine metric is given by $g = e^{\psi}|dz|^2$ for some function ψ. Parametrize the surface by

$$f : \mathcal{D} \to \mathbb{R}^3, \tag{6.1}$$

with $\mathcal{D}$ a domain in $\mathbb{C}$. Then we have the following structure equations for the affine sphere:

$$\begin{aligned} D_X Y &= \nabla_X Y + g(X, Y) f \\ D_X f &= X \end{aligned}$$

Here D is the canonical flat connection on $\mathbb{R}^3$, ∇ is a projectively flat connection, and g is the affine metric. Consider the complexified frame for the tangent bundle to the surface given by $\{e_1 = f_z = f_*(\frac{\partial}{\partial z}), e_{\overline{1}} = f_{\overline{z}} = f_*(\frac{\partial}{\partial \overline{z}})\}$.

Then we have

$$g(f_z, f_z) = g(f_{\overline{z}}, f_{\overline{z}}) = 0,\ g(f_z, f_{\overline{z}}) = \frac{1}{2} e^{\psi}.$$

Consider $\widehat{\theta}, \theta$ the matrix of connection one-forms for the Levi-Civita connection, Blaschke connection respectively. By the above equation,

$$\widehat{\theta}_{\overline{1}}^1 = \widehat{\theta}_1^{\overline{1}} = 0,\ \widehat{\theta}_1^1 = \partial\psi,\ \widehat{\theta}_{\overline{1}}^{\overline{1}} = \overline{\partial}\psi.$$

The difference $\widehat{\theta} - \theta$ is given by the Pick form J. We have

$$\widehat{\theta}_i^j - \theta_i^j = C_{ik}^j \rho^k,$$

where $\{\rho^1 = dz, \rho^{\overline{1}} = d\overline{z}\}$ is the dual frame of one-forms. With the property of the affine normal ξ such that $det(f_z, f_{\overline{z}}, \xi) = \frac{1}{2} i e^{\phi}$ and the totally symmetry of C_{ijk} in affine differential geometry (where $C_{ijk} = C_{ij}^l g_{lk}$, i.e., lower index), this determines θ:

$$\begin{pmatrix}\theta_1^1 & \theta_{\overline{1}}^1\\ \theta_1^{\overline{1}} & \theta_{\overline{1}}^{\overline{1}}\end{pmatrix}=\begin{pmatrix}\partial\psi & C_{\overline{1}\overline{1}}^1 d\overline{z}\\ C_{11}^{\overline{1}}dz & \overline{\partial}\psi\end{pmatrix}=\begin{pmatrix}\partial\psi & \overline{U}e^{-\psi}d\overline{z}\\ Ue^{-\psi}dz & \overline{\partial}\psi\end{pmatrix},$$

where we define $U=C_{11}^{\overline{1}}e^{\psi}$.

Recall that D is the canonical flat connection induced from $\mathbb{R}^3$. We get the structure equations

$$\begin{aligned}f_{zz} &= \psi_z f_z + Ue^{-\psi}f_{\overline{z}}\\ f_{\overline{z}\overline{z}} &= \overline{U}e^{-\psi}f_z+\psi_{\overline{z}}f_{\overline{z}}\\ f_{z\overline{z}} &= \frac{1}{2}e^{\psi}f\end{aligned}$$

Then, these three equations form a linear first-order system of partial differential equations in $f, f_z, f_{\overline{z}}$, where we may write as

$$\frac{\partial}{\partial z}\begin{pmatrix}f\\ f_z\\ f_{\overline{z}}\end{pmatrix}=\begin{pmatrix}0&1&0\\ 0&\psi_z&Ue^{-\psi}\\ \frac{1}{2}e^{\psi}&0&0\end{pmatrix}\begin{pmatrix}f\\ f_z\\ f_{\overline{z}}\end{pmatrix},$$

$$\frac{\partial}{\partial \overline{z}}\begin{pmatrix}f\\ f_z\\ f_{\overline{z}}\end{pmatrix}=\begin{pmatrix}0&0&1\\ \frac{1}{2}e^{\psi}&0&0\\ 0&\overline{U}e^{-\psi}&\psi_{\overline{z}}\end{pmatrix}\begin{pmatrix}f\\ f_z\\ f_{\overline{z}}\end{pmatrix}.$$

In order to have a solution of the system, the only condition is that mixed partials must commute (by the Frobenius theorem). Thus we require the two conditions

$$\begin{aligned}\psi_{zz}+U\overline{U}e^{-2\psi}-\frac{1}{2}e^{\psi} &= 0,\\ U_{\overline{z}} &= 0.\end{aligned}$$

By the above argument, we actually construct a map from the space of convex $\mathbb{R}P^2$-structures on S to the space of pairs (Σ, U). The converse direction is mainly by showing the existence and uniqueness of solutions to the above elliptic equation for a given pair (Σ, U). Combining these two directions together, we obtain the following main theorem.

MainTheorem. *(Loftin* [5]*, Labourie* [4]*) Let S be a compact oriented surface of genus $g>1$. There exists a natural bijective correspondence betwen convex $\mathbb{R}P^2$-structures on S and pairs (Σ, U), where Σ is a Riemann surface homeomorphic to S, and U is a holomorphic cubic differential on Σ.*

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A GEOMETRIC DESCRIPTION OF THE $\mathrm{PSL}_4(\mathbb{R})$-HITCHIN COMPONENT

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ABSTRACT. These notes form a rough outline of the correspondence between the $\mathrm{PSL}_4(\mathbb{R})$-Hitchin component and convex foliated projective structures from [3].

1. Introduction

Let Σ be an orientable hyperbolic surface. The *Teichmüller space* of Σ, denoted $\mathcal{T}^2(\Sigma)$, consists of conjugacy classes of discrete and faithful representations from $\Gamma := \pi_1(\Sigma)$ into $\mathrm{PSL}_2(\mathbb{R})$. This set is well known to have the topology of a ball of dimension $3\,|\chi(\Sigma)|$. There is a well known unique irreducible representation, ρ_n, from $\mathrm{PSL}_2(\mathbb{R})$ into $\mathrm{PSL_n}(\mathbb{R})$ coming from the natural action of $\mathrm{PSL}_2(\mathbb{R})$ on the symmetric product, $\mathrm{Sym}^{n-1}(\mathbb{R}^2) \cong \mathbb{R}^n$. In his article [6], Hitchin showed that the the component of the space of conjugacy classes, $\mathfrak{X}(\Gamma, \mathrm{PSL_n}(\mathbb{R}))$, containing the image of $\mathcal{T}^2(\Sigma)$ under ρ_n is also a ball of dimension $(n^2 - 1)\,|\chi(\Sigma)|$. This component is typically referred to as the *Hitchin component* and we will denote it as $\mathcal{T}^n(\Sigma)$.

For small values of n these Hitchin component can be thought of as moduli spaces of geometric structures on manifolds. For $n = 2$, the Hitchin component is the same as Teichmüller space which allows us to identify $\mathcal{T}^2(\Sigma)$ with the space of marked hyperbolic structures on Σ. More specifically, let $[\rho] \in \mathcal{T}^2(\Sigma)$. Since $\mathrm{PSL}_2(\mathbb{R})$ can be identified with the orientation preserving isometry group of $\mathbb{H}^2$ we can identify the quotient $\mathbb{H}^2/\rho(\Gamma) \cong \Sigma$, thus giving a marked hyperbolic structure on Σ. Conversely, the holonomy representation of any hyperbolic structure is a discrete and faithful representation from Γ into $\mathrm{PSL}_2(\mathbb{R})$, and equivalent marked hyperbolic structures have conjugate holonomy.

When $n = 3$, work of Goldman [2] and Goldman-Choi [1] show that the space $\mathcal{T}^3(\Sigma)$ can be identified with the moduli space of marked convex projective structures on Σ. Roughly speaking, a convex projective structure on Σ is a realization of Σ as $\Omega/\rho(\Gamma)$, where Ω is a convex set that sits inside of an affine patch (see Exercise 1) of $\mathbb{RP}^2$ and $\rho : \Gamma \to \mathrm{PSL}_3(\mathbb{R})$ is a discrete and faithful representation. For example, when ρ factors through $\mathrm{PSL}_2(\mathbb{R})$ the set Ω can be taken to be the unit disk in $\mathbb{R}^2 \subset \mathbb{RP}^2$ (See Exercise 2). While it is easy to associate a conjugacy class of discrete faithful representations with a marked projective structure, it is much more difficult to start with a representation and find an appropriate convex projective structure.

Exercise 1. *Let $\mathbb{RP}^n$ be the quotient of $\mathbb{R}^{n+1}\backslash\{0\}$ by the action of $\mathbb{R}^\times$ by scaling and let H be a hyperplane $\mathbb{RP}^n$. $\mathbb{RP}^n\backslash H$ is called an* affine patch*. The purpose of this exercise is to justify this terminology.*

(1) *Show that an affine patch can be identified with the affine space $\mathbb{R}^n$.*

(2) *Show that the subgroup* $\mathrm{PGL_{n+1}}(\mathbb{R})$ *consisting of elements that preserve* H *is equivariantly isomorphic (with respect to the identification from part (1)) to the affine group of matrices of the form*

$$\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix},$$

where $A \in \mathrm{GL}_n(\mathbb{R})$ *and* $b \in \mathbb{R}^n$.

Exercise 2. *Let* $\langle x, y\rangle = x_1y_1 + x_2y_2 - x_3y_3$ *be the standard bilinear form of signature* $(2, 1)$ *on* $\mathbb{R}^3$*. Let* $C = \{x \in \mathbb{R}^3 | \langle x, x\rangle < 0\}$ *be the cone of vectors with negative self pairing. The image,* $P(C)$*, of* C *in* $\mathbb{RP}^2$ *serves as a model for* $\mathbb{H}^2$.

(1) *Let* H *be the image of the* x_1x_2-*plane in* $\mathbb{RP}^2$*. Show that in the affine patch defined by* H *that* $P(C)$ *can be identified with the unit disk. This model is know as the* Klein Model
(2) *Let* K *be the image of the plane* $x_3 - x_2 = 0$ *in* $\mathbb{RP}^2$*. Show that in the affine patch defined by that* $P(C)$ *can be identified with the set* $v > u^2$*. (Hint, use coordinates* $u = x_1$*,* $v = x_3 + x_2$*,* $w = x_3 - x_2$ *and use inhomogeneous coordinates* $w = 1$*). This model is known as the paraboloid model.*

Our goal in this lecture to explain a correspondence between the space $\mathcal{T}^4(\Sigma)$ and certain types of projective structures on the unit tangent bundle, $S\Sigma$, of Σ. This result is originally due to Guichard and Wienhard [3]. The rough idea is that there is an $\mathbb{R}$ action on $S\Sigma$ given by the geodesic flow. This flow gives rise to a pair of foliations, $\mathcal{F}$ and $\mathcal{G}$, which are referred to as the *stable foliation* and *geodesic foliation*, respectively. We will show that representations in $\mathcal{T}^4(\Sigma)$ correspond to projective structures in which these foliations can be realized in a geometrically meaningful way inside of $\mathbb{RP}^3$. When $[\rho] \in \mathcal{T}^4(\Sigma)$ factors through $\mathrm{PSL}_2(\mathbb{R})$ then we can regard these projective structures as projective realizations of the familiar $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ structures on these manifolds.

2. $S\Sigma$ and the Geodesic Flow

In this section we will discuss important properties of $M := S\Sigma$ and the action of the geodesic flow. Let $\overline{\Gamma} = \pi_1(M)$, then $\overline{\Gamma}$ is a central (non-split) extension of Γ that fits into the following short exact sequence.

$$0 \to \mathbb{Z} \to \overline{\Gamma} \to \Gamma \to 1. \tag{2.1}$$

The manifold M has an important regular cover, $\overline{M}$, corresponding from the $\mathbb{Z}$ subgroup in (2.1). Furthermore, the foliations $\mathcal{F}$ and $\mathcal{G}$ lift to foliations $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$ of $\overline{M}$. The foliations $\mathcal{F}$ and $\mathcal{G}$ can also be lifted to the universal cover, $\tilde{M}$. These lifts will be denoted $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{G}}$, respectively. We now give two descriptions of $\overline{M}$.

Let $[\rho] \in \mathcal{T}^2(\Sigma)$ be a (conjugacy class of) representation. Choose and identification of $\tilde{\Sigma}$ with $\mathbb{H}^2$ that is equivariant with respect to ρ, such an identification is called a *uniformization*. The uniformization allows us to equivariantly identify $\overline{M}$ with the unit tangent bundle $S\mathbb{H}^2$. With this in mind we can think of a leaf of $\overline{\mathcal{G}}$ as an oriented geodesic in $\mathbb{H}^2$ and a leaf of $\overline{\mathcal{F}}$ as the union of all oriented geodesics with a common positive endpoint on $\partial\mathbb{H}^2$. Since

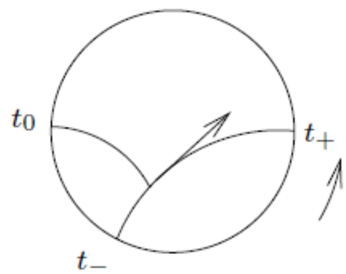

FIGURE 1. The identification between $\overline{M}$ and $\partial\Gamma^{3+}$. Picture from [3]

$\mathrm{PSL}_2(\mathbb{R})$ acts simply transitively on $S\mathbb{H}^2$ we can further identify $\overline{M}$ with $\mathrm{PSL}_2(\mathbb{R})$. Under this identification we see that $M = \rho(\Gamma)\backslash\mathrm{PSL}_2(\mathbb{R})$. Additionally, if we let

$$A = \left\{ \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \right\}, \text{ and } P = \left\{ \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} \right\},$$

be *Cartan* and *parabolic* subgroups of $\mathrm{PSL}_2(\mathbb{R})$, then the leaves of the geodesic and stable foliations can be identified with right orbits of A and P, respectively.

The uniformization also allows us to identify the boundary of Γ, $\partial\Gamma$, with $\partial\mathbb{H}^2 \cong \mathbb{RP}^1$. Furthermore, this identification gives an action of Γ on $\partial\Gamma$ that is ρ-equivariant. This boundary gives us another way to describe $\overline{M}$, $\overline{\mathcal{F}}$, and $\overline{\mathcal{G}}$. $\partial\Gamma$ inherits an orientation coming from the orientation on $\mathbb{H}^2$ and we can identify $\overline{M}$ with the set $\partial\Gamma^{3+}$ of pairwise distinct, positively oriented triples of points in $\partial\Gamma$. To see this observe that a point of $\overline{M}$ can be thought of as an oriented geodesic, L, in $\mathbb{H}^2$ and a point, x, on L. Such a point can be identified with (t_+, t_0, t_-), where t_+ and t_- are the positive and negative endpoints of L, respectively, and t_0 is the endpoint of the geodesic that intersects L perpendicularly and passes through x and makes the above triple positively oriented (see Figure 1). This identification is also equivariant with respect to the action of Γ and allows us to identify $\overline{\mathcal{F}}$ with $\partial\Gamma$ and $\overline{\mathcal{G}}$ with $\partial\Gamma^{(2)} := \partial\Gamma^2\backslash\Delta$.

3. CONVEX FOLIATED PROJECTIVE STRUCTURES

In this section we describe the geometric structures whose moduli space $\mathcal{T}^4(\Sigma)$ describes. We begin with a description of projective structures. Roughly speaking, a projective structure on a manifold is a way to locally identify the manifold with $\mathbb{RP}^n$ in such a way that the transition functions are locally elements of $\mathrm{PGL}_{n+1}(\mathbb{R})$. As such projective structures can be described in terms of atlases of charts. However, we will take a more global (but equivalent) point of view in our definition. Let N be an $n-$manifold. A *projective structure* consists of a pair (dev, hol), where $hol : \pi_1(N) \to \mathrm{PGL}_{n+1}(\mathbb{R})$ is a representation and dev is a $hol-$equivariant local homeomorphism from $\tilde{N}$ to $\mathbb{RP}^n$. Furthermore, we say that (dev_1, hol_1) and (dev_2, hol_2) are *equivalent* projective structures on N if there exists a homeomorphism $h : N \to N$ that is isotopic to the identity and an element $g \in \mathrm{PGL}_{n+1}(\mathbb{R})$ such that

- $dev_1 \circ \tilde{h} = g \circ dev_2$, where $\tilde{h}$ is a lift of h to $\tilde{N}$, and
- $hol_2 = g^{-1} hol_1 g$.

Let $\mathcal{P}(N)$ be the set of equivalence classes of projective structures on N. The above discussion shows that we have a map

$$hol : \mathcal{P}(N) \to \mathfrak{X}(\pi_1(N), \mathrm{PGL_{n+1}}(\mathbb{R})).$$

The map *dev* is called the *developing map* of the structure and the representation *hol* is called the *holonomy* of the structure.

As we mentioned before we are interested in projective structures that play well with the foliations coming from the geodesic flow. With this in mind, we say that a projective structure, (dev, hol), on M is *foliated* if the following conditions are satisfied.

- For each leaf $\tilde{g} \in \tilde{\mathcal{G}}$, $dev(\tilde{g})$ is contained in a projective line, and
- For each leaf $\tilde{f} \in \tilde{\mathcal{F}}$, $dev(\tilde{f})$ is contained in a projective plane.

Two foliated projective structures are *equivalent* if they are equivalent as projective structures and the map $h : M \to M$ preserves the foliations $\mathcal{F}$ and $\mathcal{G}$. We denote the set of equivalence classes of foliated projective structures by $\mathcal{P}_f(M)$. We now further refine this notion in order to arrive at the correct geometric structures. Let $C \subset \mathbb{RP}^n$, then C is *convex* if its intersection with every projective line is connected. If C is a convex subset of $\mathbb{RP}^n$ then C is *properly convex* if its closure does not contain a affine line.

Exercise 3. *Show that a subset of $\mathbb{RP}^n$ is properly convex if and only if its closure is contained in an affine patch.*

We can now define the appropriate projective structures. We say that a foliated projective structure on M is *convex* if the image of each leaf of $\tilde{\mathcal{F}}$ under the developing map is a convex set of a projective plane. Additionally, we define a *properly convex* foliated projective structure on M to be a foliated projective structure for which the image of each leaf of $\tilde{\mathcal{F}}$ is mapped to a properly convex subset of a projective plane by the developing map. Let $\mathcal{P}_{pcf}(M)$ subset of $\mathcal{P}_f(M)$ consisting of equivalence classes of properly convex foliated projective structures.

We can now rephrase the correspondence between $\mathcal{T}^4(\Sigma)$ and projective structures in more precise terms. Let $p : \overline{\Gamma} \to \Gamma$ be the projection implicit in (2.1). The map p gives an embedding of $\mathcal{T}^4(\Sigma) \subset \mathfrak{X}(\Gamma, \mathrm{PSL}_4(\mathbb{R})) \subset \mathfrak{X}(\overline{\Gamma}, \mathrm{PSL}_4(\mathbb{R}))$, and the correspondence can be succinctly stated as

Proposition 3.1. *The map hol is a homeomorphism between $\mathcal{P}_{pcf}(M)$ and $\mathcal{T}^4(\Sigma)$.*

Remark 3.2. *Since Γ has trivial center* (2.1) *implies that the center of $\overline{\Gamma}$ is cyclic, and we denote its generator by τ. The above correspondence implies that the holonomy of a properly convex foliated projective structure on M factors through p and thus every such holonomy kills τ.*

4. Examples and Ideas

In this section we will discuss certain examples of properly convex foliated projective structures on M and discuss some of the ideas required to prove Proposition 3.1. Let

$[\rho] \in \mathcal{T}^2(\Sigma)$, then we can define an element of $\mathcal{P}_{pcf}(M)$ as follows. Let $[Q] \in \mathbb{RP}^3$, where $Q = x(x^2+y^2)$ (here we are using the fact that $\mathbb{R}^4 \cong \mathrm{Sym}^3(\mathbb{R}^2)$). Using the fact that $\overline{M} \cong \mathrm{PSL}_2(\mathbb{R})$ we can define a projective structure by letting dev[1] be the map $g \mapsto \rho_4(\rho(g)) \cdot [Q]$, where $g \in \mathrm{PSL}_2(\mathbb{R})$ and letting $hol = \rho_4 \circ \rho$. The vector $Q = (1,0,1,0)$ in the standard basis for $\mathrm{Sym}^3(\mathbb{R}^2)$ and so

$$\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \mapsto \begin{pmatrix} e^{3t/2} & 0 & 0 & 0 \\ 0 & e^{t/2} & 0 & 0 \\ 0 & 0 & e^{-t/2} & 0 \\ 0 & 0 & 0 & e^{-3t/2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{3t/t} \\ 0 \\ e^{-t/2} \\ 0 \end{pmatrix} = \begin{pmatrix} e^{2t} \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

and

$$\begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} \mapsto \begin{pmatrix} a^3 & a^2b & ab^2 & b^3 \\ 0 & a & 2b & 3b^2/a \\ 0 & 0 & 1/a & 3b/a^2 \\ 0 & 0 & 0 & 1/a^3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a^3+ab^2 \\ 2b \\ 1/a \\ 0 \end{pmatrix} = \begin{pmatrix} a^2(a^2+b^2) \\ 2ab \\ 1 \\ 0 \end{pmatrix}.$$

Under the coordinate change $v = a^2(a^2+b^2)$, $u = 2ab$, we see that the above image can be identified with $v > u^2/4$, and is thus properly convex (See Exercise 3). In fact, this set can be identified with $\mathbb{H}^2$ (see Exercise 2). Thus we see that this projective structure is properly convex foliated.

Despite knowing that these structures are properly convex foliated, we do not have a very good idea of what they look like globally. In order to get a more global picture we will try to understand the developing map in terms of the description of $\overline{M}$ as $\partial\Gamma^{3+}$. Let V be a vector space and let $Flag(V)$ denote the flag variety of V. If we think of $\mathbb{R}^2$ as $\mathrm{Sym}^1(\mathbb{R}^2)$, then the *Veronese embedding* gives the following equivariant curve $\xi : \partial\Gamma \cong \mathbb{RP}^1 \to Flag(\mathbb{R}^4)$. Given by $\xi = (\xi^1, \xi^2, \xi^3)$, where

- $\xi^1([S])$ is the line of polynomials divisible by S^3,
- $\xi^2([S])$ is the plane of polynomials divisible by S^2, and
- $\xi^3([S])$ is the hyperplane of polynomials divisible by S.

If we let Ω_ξ be the set of polynomials in $\mathbb{RP}^3$ with a single real root (i.e. they factor over $\mathbb{R}$ into a linear and a quadratic term), then Ω_ξ is the image of dev.

Exercise 4. *Prove that Ω_ξ is the image of dev, namely that Ω_ξ is the* $\mathrm{PSL}_2(\mathbb{R})$ *orbit of* $[Q]$ *under the action $g \cdot [R] = \rho_4(\rho(g)) \cdot [R]$.*

The map ξ allows us to define a family, ξ^1_t of equivariant maps from $\partial\Gamma \to \mathbb{RP}^3$, given by

$$\xi^1_t(t') = \begin{cases} \xi^3(t) \cap \xi^2(t') & \text{if } t \neq t' \\ \xi^1(t) & \text{if } t = t' \end{cases}$$

For each t, the image of ξ^1_t in $\xi^3(t)$ bounds the copy of $\mathbb{H}^2_t$ given by $dev(t)$ (here we are thinking of t as a leaf of $\overline{\mathcal{F}}$). The geodesic leaf $g = (t_+, t_-)$ is taken to the intersection of $\mathbb{H}^2_t$ and the projective line, $\overline{\xi^1(t_+)\xi^1_{t_+}(t_-)}$, connecting $\xi^1(t_+)$ and $\xi^1_{t_+}(t_-)$. The tangent lines to $\mathbb{H}^2_t$ at $\xi^1(t_+)$ and $\xi^1_{t_+}(t_-)$ are $\xi^2(t_+)$ and $\xi^3(t_+) \cap \xi^3(t_-)$, respectively. Furthermore, these

[1]Technically, dev is a lift of this map to $\tilde{M}$.

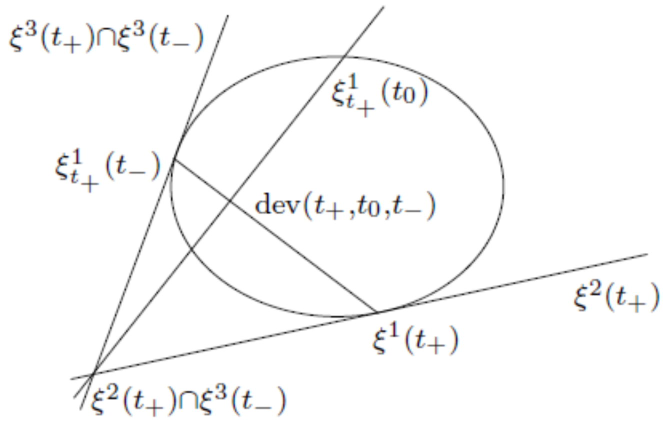

FIGURE 2. The image of the developing map. Picture from [3].

lines intersect in the point $\xi^3(t_-)\cap\xi^2(t_+)=\xi^1_{t_-}(t_+)$. Finally, given t_0 different from t_+ and t_- there is a unique point of intersection between $\overline{\xi^1(t_+)\xi^1_{t_+}(t_-)}$ and $\overline{\xi^1_{t_-}(t_+)\xi^1_{t_+}(t_0)}$. Thus we see that in terms of $\partial\Gamma^{3+}$ that *dev* is defined by

$$(t_+,t_0,t_-)\mapsto \overline{\xi^1(t_+)\xi^1_{t_+}(t_-)}\cap\overline{\xi^1_{t_-}(t_+)\xi^1_{t_+}(t_0)}. \tag{4.1}$$

The discussion of the previous paragraph is illustrated in Figure 2

5. Convex Representations

The proof of Proposition 3.1 relies on the fact that representations in $\mathcal{T}^4(\Sigma)$ can be characterized by a certain convexity property which we now discuss. Work of Labourie and Guichard [5, 7] has shown that $\rho\in\mathcal{T}^4(\Sigma)$ if and only if ρ is a convex representation. A representation is *convex* if we can find a ρ-equivariant curve $\xi^1:\partial\Gamma\to\mathbb{RP}^3$ such that for $t_1,\dots t_4$ that are pairwise distinct,

$$\xi^1(t_1)\oplus\ldots\oplus\xi^1(t_4)=\mathbb{R}^4.$$

A simple exercise shows that a convex curve in $\mathbb{RP}^2$ bounds a properly convex set. Additionally, a curve $\xi=(\xi^1,\dots,\xi^{n-1}):\mathbb{RP}^1\to Flag(\mathbb{R}^n)$ is called *Frenet* (this is sometimes known as hyperconvex) if

(1) For every $(n_1,\dots,n_k)$ such that $\sum_{i=1}^k n_i=n$ and every $t_1,\dots,t_k\in\mathbb{RP}^1$ of pairwise distinct elements the following sum is direct

$$\sum_{i=1}^k \xi^{n_i}(x_i)=\mathbb{R}^n.$$

(2) For every $(m_1, \ldots, m_k)$ with $\sum_{i=1}^k m_i = m \leq n$ and every $x \in \mathbb{RP}^1$

$$\lim_{(x_i)\to x} \sum_{i=1}^{k} \xi^{m_i}(x_i) = \xi^m(x),$$

where the limit is taken over $k-$tuples of pairwise distinct points.

It is easy to see that if ξ is Frenet then ξ^1 is a convex curve and that ξ^1 determines ξ via the limit condition. By work of Labourie [7] has shown that If a ρ is convex then it is possible to find a unique ρ-equivariant Frenet curve. Given a non-degenerate bilinear form on $\mathbb{R}^n$ and a curve $f = (f^1, \ldots, f^{n-1}) : \mathbb{RP}^1 \to Flag(\mathbb{R}^n)$, it is possible to define a dual curve $f^\perp = (f^{n-1,\perp}, \ldots, f^{1,\perp}) : \mathbb{RP}^1 \to Flag(\mathbb{R}^{n*})$. Work of Guichard [4] shows that a curve ξ is Frenet if and only if $\xi^\perp$ is Frenet. This duality will be crucial in the proof of Proposition 3.1.

5.1. **convex implies properly convex foliated.** We begin by showing that an element $[\rho] \in \mathcal{T}^4(\Sigma)$ gives rise to a properly convex foliated projective structure on M. By the previous paragraph we see that ρ is a convex representation and thus we can find a ρ-equivariant flag curve ξ. We begin by using this curve to define a family of ρ-equivariant lower dimensional flag curves. Define $\xi_t : \mathbb{RP}^1 \to Flag(\xi^3(t))$ by

$$\xi_t(t') = \begin{cases} (\xi^3(t) \cap \xi^2(t'), \xi^3(t) \cap \xi^3(t')) & \text{if } t \neq t' \\ (\xi^1(t), \xi^2(t)) & \text{if } t = t' \end{cases} \tag{5.1}$$

For each $t \in \mathbb{RP}^1$ the curve ξ_t is also Frenet. This is proven by showing that $\xi_t^\perp$ is Frenet and using the basic fact that if W and V are linear subspace then $(V+W)^\perp = V^\perp \cap W^\perp$. For example, to show that $\xi_t^\perp$ is Frenet one of the things we need to show that $(\xi_t^{2,\perp}(t_1) + \xi_t^{1,\perp}(t_2)) = \mathbb{R}^3$ for all distinct pairs t_1, t_2. To show this we observe that

$$(\xi_t^{2,\perp}(t_1)+\xi_t^{1,\perp}(t_2))^\perp = \xi_t^2(t_1)\cap\xi_t^1(t_2) = \xi^3(t)\cap\xi^3(t_1)\cap\xi^2(t_2) = (\xi^{3,\perp}(t)+\xi^{3,\perp}(t_1)+\xi^{2,\perp}(t_2))^\perp = \{0\},$$

with the last equality coming from the fact that $\xi^\perp$ is Frenet. The other conditions needed to be verified to show that $\xi^\perp$ is Frenet can all be shown in a similar way. We now define a developing map using the formula in (4.1). For each t the image of ξ_t^1 bounds is convex and thus bounds a properly convex subset C_t of $\xi^3(t)$. The Frenet properties of ξ this new developing map has all the same nice properties as the map given to us in the previous example by the Veronese embedding.

5.2. **properly convex foliated implies convex.** The more difficult direction is to show that given a properly convex foliated projective structure on M that the holonomy representation is a convex representation. Details can be found in [3] and we simply outline the key ideas. Suppose that we have such a structure with holonomy ρ. Since we know that the image of a leaf of $\tilde{\mathcal{F}} \cong \tilde{\Gamma}$ under the developing map is contained in a projective plane we get a map[2] $\xi^3 : \partial\tilde{\Gamma} \to \mathbb{RP}^{3*}$taking $t \in \partial\tilde{\Gamma}$ to the projective plane containing $dev(t)$. The first thing that we have to do is to show that the map ξ^3 descends to a map defined on $\partial\Gamma$ (this

[2]Here we are implicitly identifying the space of projective planes in $\mathbb{RP}^3$ with $\mathbb{RP}^{3*}$ using a non-degenerate bilinear form.

part is highly non-trivial and comprises a good chunk of [3]). We must then show that this map is convex. Namely that for $t_1, \ldots, t_4$ pairwise distinct points that

$$\xi^3(t_1) + \ldots + \xi^3(t_4) = \mathbb{R}^{4*},$$

or equivalently that

$$\xi^3(t_1) \cap \ldots \cap \xi^3(t_4) = \emptyset. \tag{5.2}$$

The fact that these planes do not have a common intersection can be viewed geometrically. Fix t_1, then the fact that the intersection from (5.2) is empty is equivalent to the three lines lines $\xi^3(t_1) \cap \xi^3(t_i)$, $2 \leq i \leq 4$, not intersecting. Let C_{t_1} be the properly convex set that is the image of the developing map restricted to plane $\xi^3(t_1)$. Then it can be shown (with a good deal of work) that the domain C_t is *strictly convex* (contains no line segments in its boundary) and that the lines $\xi^3(t_1) \cap \xi^3(t_i)$ for $2 \leq i \leq 4$ are tangent lines to C_{t_1} at distinct points, and thus do not intersect. Try drawing tangents to the domain in Figure 2 to convince yourself that these lines must be disjoint.

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