A PRIMER ON COHOMOLOGICAL METHODS IN REPRESENTATION THEORY OF SURFACE GROUPS

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1. Introduction

These are extended notes from a series of talks that the author delivered in a workshop on Higher Teichmüller Thurston Theory in Northport, Maine in July 2013. The main story covered in this series of talks is centered around the following three aspects:

- We want to introduce some *basic cohomological tools*, which are useful in general representation theory of groups (and beyond). In particular, we discuss Gromov-Traubel-Johnson's *bounded cohomology* of discrete groups.
- We give one particularly simple (but in some sense prototypical) example which illustrates how cohomological tools can be used to study representations of surface groups. Namely, we give a purely cohomological condition (taken from [1]), which ensures that a given representation of a surface group into a Lie group is *faithful* and has *discrete image*.
- We explain how various geometric and/or topological assumptions (maximal Toledo invariant, existence of an Anosov structure over the Shilov boundary) imply that a representation is discrete and faithful by *exploiting this cohomological criterion.* Thereby we prove some simple special cases of results from [8] and [1].

We emphasize that there are many other things about representations besides discreteness and faithfulness that one can study using cohomological tools, even for surface groups; however, for limitations of time and space we will mostly focus on these two properties.

We have decided to present the material in a systematic rather than in a historical order. Let us point out right from the beginning that none of the results presented here are new, albeit a few details are presented differently than in the standard literature. All the material concerning group cohomology presented here is completely standard and can be found in any textbook on the subject, see e.g. [2]. Since it is easily accessible, we give basically no proofs. The modern theory of bounded group cohomology is not yet completely accessible in textbook form (although [12] covers some major parts of it). The two dissertations [29, 3] provide a good overview. Classical sources include [27, 23, 26, 22].

The whole uncensored story how bounded cohomology can be used to study representations of surface group is told in the recent survey [5], which covers in particular the breakthrough results from [8]. The present note was written with an eye towards that survey. This means that almost everything we say here, is also said in

[5] in greater generality. In particular, while we restrict ourselves to closed surfaces, [5] treats also the case of surfaces with boundary. It is our hope that the present notes can serve as an elementary introduction to [5]. Concerning bibliographical references we strongly recommend the reader to take a look at [5], since we will not be able to discuss or even repeat the more than 100 references here. It is not quite clear to the author what the historical starting point of this whole circle of ideas actually is, but Milnor's famous paper [28] was certainly a major early inspiration.

Let us mention explicitly the few places in which we deviate from [5]. Firstly and most importantly, in proving discreteness and faithfulness of maximal representations we make us of a very general criterion, which was established only recently in [1]. This allows us to reprove at the same time discreteness and faithfulness of certain Anosov representations. The corresponding part of these notes is taken almost literally from [1]. Our second deviation concerns the treatment of the bounded Euler class, which is one of the major tools in the cohomological study of surface groups. A famous theorem of Ghys says that it is a complete quasi-conjugacy invariant of representations, and this theorem can be invoked to detect trivial subgroups of surface groups using bounded cohomology. However, what is actually needed in this argument is only a very special case of Ghys' theorem, and we present here an elementary proof of this special case due to Michelle Bucher and the author. For the reader interested in the full story we recommend Ghys' original papers [19, 20]. Our third deviation from the standard literature concerns our treatment of the second bounded cohomology. We show that second bounded cohomology classes can be identified with equivalence classes of certain objects that we call *quasi-corners*, and which generalize central extensions and quasimorphisms at the same time. We find this language very useful from a pedagogical point of view and believe that it helps to clarify some statements, but it is essentially just a reformulation of very old ideas (going all the way back to Poincaré).

Finally, some remarks concerning prerequisites. These talks were given in front of a very specific audience, which was familiar with representations of surface groups in general and Anosov representations in particular, but not necessarily with group cohomology. We therefore develop all cohomological tools from scratch (some without proof), while we take various basic facts in hyperbolic geometry for granted. Probably our greatest omission is that we do not discuss the definition of Anosov representations at all. However, the reader who wants to close this gap finds more than sufficient information in the contributions of Spencer Dowdall and Tengren Zhang in this collection [17, 31]. The odd sections of the body of this article (i.e. 3, 5 and 7) do not refer to surface groups or representations in any essential way (except for examples), and provide an introduction to bounded cohomology which might be of independent interest. However, we warn the reader that this introduction is heavily biased by the applications we have in mind. In particular, we focus completely on (bounded) cohomology in degree 2.

Throughout these notes we restrict attention to closed oriented surfaces. This is morally wrong, since the true power of the bounded cohomology machinery only becomes visible in the case of surfaces with boundary. However, since our main goal is to be elementary this restriction was, unfortunately, unavoidable. We encourage the reader to read the uncensored story in [5].

Contents

2. Reminder of group cohomology

In this section we recall some basic facts from group cohomology. We refer to the textbook literature for proofs. What we actually need is rather little, so any textbook on group cohomology such as [2] will do.

2.1. **What is group cohomology?** In textbooks on homological algebra, group cohomology is often introduced as follows:

Definition 2.1. Let G be a group, R a ring and $RG-\mathfrak{Mod}$ the category of left RG-modules. Let

$$(-)^G : RG - \mathfrak{Mod} \to R - \mathfrak{Mod}, \quad V \mapsto V^G := \{v \in V \,|\, g.v = v\}$$

be the functor of G-invariants. Denote by

$$H^n(G; -) := \mathcal{D}^n(-)^G : RG - \mathfrak{Mod} \to \mathfrak{Mod}$$

its n-th derived functor. Then $H^n(G;V)$ is called the n-th group cohomology of G with coefficients in V.

What does this mean?

- Firstly, for every RG-module V there is a certain R-module $H^n(G;V)$, and the assignment $V \mapsto H^n(G;V)$ is functorial.
- In order to compute the isomorphism class of $H^n(G;V)$ one proceeds as follows: One first chooses an augmented co-resolution

  $$0 \to V \to I^0 \to I^1 \to I^2 \to \dots$$

  of V by injective RG-modules. One then deletes the augmentation and applies the functor $(-)^G$ to obtain a co-complex

  $$0 \to (I^0)^G \to (I^1)^G \to (I^2)^G \to \dots$$

  The cohomology of this cocomplex is then precisely $H^\bullet(G;V)$.

If these words don't mean anything to you, here is what you should remember: The invariants $H^\bullet(G;V)$, which depend only on G and V can be computed from many different resolutions. Therefore the same cohomology class can take very different meanings (geometric/topological/algebraic) in different contexts. This is the reason why group cohomology can be used to translate algebraic into geometric problems and vice versa. Fortunately, we will work with only a few very specific resolutions, which we discuss in detail in the sequel. We will be mostly interested in the case where $R \in \{\mathbb{Z}, \mathbb{R}\}$ and $V = R$ is the trivial RG-module.

2.2. **The topological resolution.** The following lemma links the purely algebraic definition of group cohomology to topology. Various version of this result were discovered independently by the American school around Eilenberg and McLane and the Swiss school around Eckmann and Hopf in the 1940s. Historically, this was the starting point of the modern theory of group cohomology.

Lemma 2.2. (i) *For every group G there exists a space BG, unique up to homotopy, such that $\pi_1(BG) = G$ and the universal covering EG of BG is contractible.*

(ii) $\mathbb{Z} \to C^0_{\text{sing}}(EG) \to C^1_{\text{sing}}(EG) \to \dots$ *is an augmented resolution of $\mathbb{Z}$.*

(iii) $H^\bullet(G; \mathbb{Z}) \cong H^\bullet(BG; \mathbb{Z})$.

Example 2.3. (i) $B\mathbb{Z} = \mathbb{S}^1$, hence

$$H^\bullet(\mathbb{Z}; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}.$$

(This is our short-hand notation to mean that $H^0(\mathbb{Z}; \mathbb{Z}) = \mathbb{Z}$, $H^1(\mathbb{Z}; \mathbb{Z}) = \mathbb{Z}$ and $H^n(\mathbb{Z}; \mathbb{Z}) = 0$ for $n > 1$.)

(ii) If $\Gamma_g := \langle a_1, b_1, \dots, a_g, b_g \mid \prod_{j=1}^g [a_j, b_j] \rangle$, then $B\Gamma_g = \Sigma_g$, the closed surface of genus g. Thus

$$H^\bullet(\Gamma_g; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}^{2g} \oplus \mathbb{Z}.$$

Historically, group cohomology was first *defined* as the cohomology of the classifying space. The usefulness of the derived functor interpretation only became apparent later on.

2.3. **The (in-)homogeneous bar resolution.** The homogeneous bar resolution is a combinatorial model for cohomology based on the injective resolution with

$$I^n := C^n(G; V) := \operatorname{Map}(G^{n+1}, V)$$

and *homogeneous differentials*

$$d : C^{n-1}(G; V) \to C^n(G; V), \quad dc(g_0, \dots, g_n) = \sum_{j=0}^n (-1)^j c(g_0, \dots, \widehat{g_j}, \dots, g_n).$$

Thus,

$$H^\bullet(G; V) = H^\bullet(C^0(G; V)^G \to C^1(G; V)^G \to C^2(G; V)^G \to C^3(G; V)^G \to \dots)$$

Note that there are isomorphisms

$$\iota_n : C^n(G; V)^G := \operatorname{Map}(G^{n+1}, V)^G \cong \operatorname{Map}(G^n; V)$$

given explicitly by

$$(\iota_n f)(g_1, \dots, g_n) = f(e, g_1, g_1g_2, \dots, g_1 \cdots g_n), \quad (\iota_n^{-1} h)(g_0, \dots, g_n) = g_0.h(g_0^{-1}g_1, g_1^{-1}g_2, \dots, g_{n-1}^{-1}g_n).$$

Thus, if we define $\partial^n := \iota_n^{-1} \circ d \circ \iota_n$, then

$$H^\bullet(G; V) = H^\bullet(\operatorname{Map}(G^0, V) \xrightarrow{\partial^0} \operatorname{Map}(G^1, V) \xrightarrow{\partial^1} \operatorname{Map}(G^2, V) \xrightarrow{\partial^2} \operatorname{Map}(G^3, V) \to \dots).$$

This is called the *inhomogeneous bar resolution.*

Exercise 2.4. (i) Give explicit formulas for $\partial^0, \partial^1, \partial^2$.

(ii) Use the inhomogeneous bar resolution to show that

$$H^0(G;V) \cong V^G,$$

and if $V = R$ is the trivial RG-module then

$$H^1(G;R) \cong \mathrm{Hom}(G,R).$$

2.4. **The second cohomology.** According to Exercise 2.4, a 2-cocycle for the inhomogeneous bar resolution of the trivial module R is a function $c : G^2 \to R$ satisfying

$$\partial^2 c(g,h,k) = c(h,k) - c(gh,k) + c(g,hk) - c(g,h) = 0.$$

A 2-coboundary is a function $c : G^2 \to R$ for which there exists $f : G \to R$ such that

$$c(g,h) = \partial^1 f(g,h) = f(h) - f(gh) + f(h).$$

What is the algebraic meaning of a cohomology class $[c] \in H^2(G;R)$?

Definition 2.5. A *central extension* of G by R is a short exact sequence of the form

$$\xi = (0 \to R \xrightarrow{i} \widetilde{G} \xrightarrow{p} G \to 1) \tag{2.1}$$

with $R < C(\widetilde{G})$, the center of $\widetilde{G}$. A *right section* σ of the central extension (2.1) is a map $\sigma : G \to \widetilde{G}$ with $p\sigma = 1_G$. Two central extensions are *equivalent* if there exists an isomorphism making the diagram

$$\begin{tikzcd}
& & \widetilde{G}_1 \arrow[dd, "\cong"] \arrow[dr] & & \\
0 \arrow[r] & R \arrow[ur] \arrow[dr] & & G \arrow[r] & 1 \\
& & \widetilde{G}_2 \arrow[ur] & &
\end{tikzcd}$$

commute.

Proposition 2.6 (Classification of central extensions)**.** (i) *Let a central extension ξ as in* (2.1) *be given and let σ be a right-section. Then the map*

$$c_\sigma(g,h) := i^{-1}(\sigma(g)^{-1}\sigma(gh)\sigma(h)^{-1})$$

is a cocycle.

(ii) *If ρ is another right-section, then the difference $c_\sigma - c_\rho$ is a coboundary. Thus the cohomology class $e(\xi) := [c_\sigma] \in H^2(G;R)$ depends only on ξ.*

(iii) *The map $\xi \mapsto e(\xi)$ is an isomorphism between the set* $\mathrm{Ext}(G;R)$ *of isomorphism classes of central extensions of G by R and $H^2(G;R)$.*

In the sequel we refer to $e(\xi)$ as the *Euler class* of the central extension ξ.

2.5. **Functoriality and the lifting obstruction.** An important property of the cohomology functors $H^\bullet(H;R)$ with trivial coefficients is that they are also (contravariantly) functorial in the group variable. Explicitly, if $\rho : H \to G$ is a homomorphism and $c \in \mathrm{Map}(G^n;R)$ is an inhomogeneous then

$$\rho^* c(h_1, \ldots, h_n) := c(\rho(h_1), \ldots, \rho(h_n))$$

defines an inhomogeneous H-cocycle, and we obtain a map

$$\rho^* : H^\bullet(G;R) \to H^\bullet(H;R), \quad [c] \mapsto [\rho^* c]$$

in the opposite(!) direction. This map can also be defined more abstractly; either way, one can show that it is compatible with the geometric resolution: If H and G are groups and $\rho : H \to G$ is a homomorphism, then there exists a ρ-equivariant map $\varphi : BH \to BG$ (unique up to homotopy) and the diagram

$$\begin{tikzcd} H^\bullet(G;R) \arrow[r, "\rho^*"] \arrow[d, "\cong"'] & H^\bullet(H;R) \arrow[d, "\cong"] \\ H^\bullet(BG;R) \arrow[r, "\varphi^*"] & H^\bullet(BH;R) \end{tikzcd}$$

commutes.

We now specialize again to degree 2: Let $e \in H^2(G;\mathbb{R})$ and $\rho : H \to G$ be a homomorphism. What is the meaning of $\rho^* e$? Recall that, by Proposition 2.6 there exists a central extension ξ (unique up to isomorphism) such that $e = e(\xi)$. Assume that ξ is given as in (2.1). We say that the homomorphism ρ *lifts* over the central extension ξ if there exists a homomorphism $\widetilde{\rho} : H \to \widetilde{G}$ making the diagram

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r] & \widetilde{G} \arrow[r] & G \arrow[r] & 1 \\ & & & H \arrow[ul, dotted, "\widetilde{\rho}"] \arrow[u, "\rho"'] & \end{tikzcd}$$

commute.

Proposition 2.7 (Lifting obstruction)**.** *Let $e = e(\xi) \in H^2(G;R)$ and $\rho : H \to G$ be a homomorphism. Then $\rho^* e = 0 \in H^2(H;R)$ if and only if ρ lifts over the extension ξ. Moreover, if $\rho^* c_\sigma = du$ for some $u : H \to R$, then an explicit lift $\widetilde{\rho}$ is given by*

$$\widetilde{\rho}(h) = \sigma(\rho(h)) \cdot i(u(h)). \tag{2.2}$$

While the abstract obstruction result is standard, the explicit formula for the lift is not always given in the literature. Since we will heavily use this formula later on, we provide a proof:

Proof. A simple computation shows that if $\rho^* c_\sigma = du$ and $\widetilde{\rho}$ is defined as in (2.2) then

$$\begin{aligned} \widetilde{\rho}(\gamma_1\gamma_2) &= \sigma(\rho(\gamma_1\gamma_2))i(u(\gamma_1\gamma_2)) = \sigma(\rho(\gamma_1)\rho(\gamma_2)) \cdot i(u(\gamma_1\gamma_2)) \\ &= \sigma(\rho(\gamma_1))i(e_\sigma(\rho(\gamma_1), \rho(\gamma_2)))\sigma(\rho(\gamma_2))i(u(\gamma_1\gamma_2)) \\ &= \sigma(\rho(\gamma_1))i(du(\gamma_1, \gamma_2))\sigma(\rho(\gamma_2))i(-u(\gamma_1\gamma_2)) \end{aligned}$$

$$
\begin{aligned}
&= \sigma(\rho(\gamma_1))i(u(\gamma_2))i(-u(\gamma_1\gamma_2))i(u(\gamma_1))\sigma(\rho(\gamma_2))i(u(\gamma_1\gamma_2))\\
&= \sigma(\rho(\gamma_1))i(u(\gamma_1))\sigma(\rho(\gamma_2))i(u(\gamma_2))\\
&= \widetilde{\rho}(\gamma_1)\widetilde{\rho}(\gamma_2),
\end{aligned}
$$

whence $\widetilde{\rho}$ is indeed a homomorphism. This shows in particular that ρ lifts whenever $\rho^*e = 0$. We leave the converse implication to the reader. □

Corollary 2.8 (Killing H^2 via central extensions)**.** *Let G be a group and $e \in H^2(G; R)$. Then there exists a central extension ξ as in* (2.1) *such that $p^*e = 0 \in H^2(\widetilde{G}; R)$.*

Proof. Choose ξ so that $e = e(\xi)$. Then

$$
\begin{tikzcd}
0 \arrow[r] & R \arrow[r] & \widetilde{G} \arrow[r, "p"] & G \arrow[r] & 1 \\
 & & & \widetilde{G} \arrow[ul, "1_{\widetilde{G}}"] \arrow[u, "p"'] &
\end{tikzcd},
$$

commutes, whence $p^*e = 0$. □

2.6. **The case of surface groups.** We now specialize the above results in the case where $G = \Gamma_g$ is a surface group. From the topological model we know that $H^2(\Gamma_g; \mathbb{Z}) \cong \mathbb{Z}$. By Proposition 2.6 there exists thus for every integer n a central $\mathbb{Z}$-extension $\widetilde{\Gamma_g^{(n)}}$ such that these exhaust all equivalence classes of central $\mathbb{Z}$-extensions. The extension $\widetilde{\Gamma_g^{(0)}}$ is just the trivial extension $\Gamma \times \mathbb{Z}$. In general we have presentations

$$
\widetilde{\Gamma_g^{(n)}} = \langle a_1, b_1, \ldots, a_g, b_g, c \,|\, \prod_{j=1}^{g} [a_j, b_j] = z^n,\ [a_j, c] = [b_j, c] = e \rangle,
$$

where the kernel of the map $\widetilde{\Gamma_g^{(n)}} \to \Gamma_g$ is generated by c. Since $H_b^2(G, \mathbb{R})$ is cyclic, a special role is played by the central extension

$$
p_{\Gamma_g} : \widetilde{\Gamma_g} \to \Gamma_g,
$$

where $\widetilde{\Gamma_g} := \widetilde{\Gamma_g^{(1)}}$. This central extension is universal in the following sense:

Proposition 2.9 (Universal central extensions of surface groups)**.** *Let $p_G : \widetilde{G} \to G$ be a central $\mathbb{Z}$-extension and $\rho : \Gamma_g \to G$ an arbitrary homomorphism. Then there exists a lift $\widetilde{\rho} : \widetilde{\Gamma_g} \to \widetilde{G}$ making the diagram*

$$
\begin{tikzcd}
\widetilde{\Gamma_g} \arrow[r, dotted, "\widetilde{\rho}"] \arrow[d, "p_{\Gamma_g}"'] & \widetilde{G} \arrow[d, "p_G"] \\
\Gamma_g \arrow[r, "\rho"] & G
\end{tikzcd}
$$

commute.

Proof. Let $\alpha \in H^2(G;\mathbb{Z})$ be the class associated with the extension p_G and $\beta := \rho^*\alpha$. Then there exists $n \in \mathbb{Z}$ such that β is represented by the extension $p_n : \widetilde{\Gamma_g^{(n)}} \to \Gamma_g$. Now there is a homorphism $\widetilde{\Gamma} \to \widetilde{\Gamma_g^{(n)}}$ given by $c \mapsto c^n$, hence we get

$$\begin{tikzcd}
\widetilde{\Gamma_g} \arrow[d] \arrow[dr, dotted, "\widetilde{\rho}"] \arrow[dd, bend right=60] & \\
\widetilde{\Gamma_g^{(n)}} \arrow[r] \arrow[d, "p_{\Gamma_g}"'] & \widetilde{G} \arrow[d, "p_n"] \\
\Gamma_g \arrow[r, "\rho"] & G.
\end{tikzcd}$$

□

2.7. **Actions on the circle.** We provide a first application of the methods developed so far. Let Γ be a group acting on the circle by orientation-preserving homomorphisms. When does this action lift to an action on the real line?

To answer this question, consider the group $G := \mathrm{Homeo}^+(S^1)$ of orientation-preserving homeomorphisms of the circle and its central extension

$$\xi_{S^1} = (0 \to \mathbb{Z} \to \widetilde{G} \to G \to 1),$$

where

$$\widetilde{G} := \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) = \{f \in \mathrm{Homeo}^+(\mathbb{R}) \mid f(x+1) = f(x)+1\}.$$

The class $e(S^1) := e(\xi_{S^1})$ is called the *Euler class of the circle*. Now an orientation-preserving action of Γ on the circle is the same as a homomorphism $\rho : \Gamma \to G$. We refer to $\rho^*(e(S^1)) \in H^2(\Gamma)$ as the Euler class of this circle action. Then we obtain the following special case of Proposition 2.7:

Corollary 2.10. *An action of Γ on the circle lifts to an action on the real line if and only if the Euler class of the circle action vanishes.*

3. Bounded cohomology I: Elementary theory

3.1. **Definition and models.** Bounded group cohomology was introduced independently by Traubel (working on groups, unpublished) and Johnson ([27], working on Banach algebras) and popularized by Gromov in his famous 1982 paper [23]. It is an important tool in modern group theory ever since. As for classical cohomology, we can give a functorial, a topological and a combinatorial definition. The combinatorial definition is the easiest to state, so we start with this one:

Recall that the homogeneous standard resolution, say with integer coefficients, is formed by the modules $C^n(G;\mathbb{Z}) := \mathrm{Map}(G^{n+1},\mathbb{Z})$ and homogeneous differentials d. If we define $C^n_b(G;\mathbb{Z}) := l^\infty(G^{n+1},\mathbb{Z})$, where $l^\infty(-;\mathbb{Z})$ denotes bounded $\mathbb{Z}$-valued functions, then $(C^n_b(G;\mathbb{Z}), d)$ is a subcomplex of $(C^n(G;\mathbb{Z}), d)$ and we define:

Definition 3.1. The cohomology $H^\bullet_b(G;\mathbb{Z}) := H^\bullet(C^n_b(G;\mathbb{Z}), d)$ is called the *bounded integral group cohomology* of G.

As in the classical case, we also have an inhomogeneous model

$$H^\bullet_b(G;\mathbb{Z}) \cong H^\bullet(l^\infty(G^0;\mathbb{Z}) \to l^\infty(G^1;\mathbb{Z}) \to l^\infty(G^2;\mathbb{Z}) \to l^\infty(G^3;\mathbb{Z}) \to \dots).$$

Bounded group cohomology with *real coefficients* is defined accordingly, replacing $\mathbb{Z}$ by $\mathbb{R}$ everywhere. Integral and real bounded group cohomology are related by the *Gersten exact sequence*

$$\dots \to H^n_b(G;\mathbb{Z}) \to H^n_b(G;\mathbb{R}) \to H^n(G;\mathbb{R}/\mathbb{Z}) \to H^{n+1}_b(G;\mathbb{Z}) \to \dots$$

We will mostly be working with real coefficients in our applications, but occasionally passing to integral coefficients is necessary.

Let us now turn to other models of bounded cohomology: One can define bounded singular cohomology of a topological space X by considering only those integral (or real-valued) cochains, which considered as functions on singular simplices are bounded. If we denote the group of this cochains by $C^n_b(X;\mathbb{Z})$ (or $C^n_b(X;\mathbb{R})$) then

$$H^\bullet_b(X;\mathbb{Z}) := H^\bullet(C^\bullet_b(X;\mathbb{Z}), d), \quad H^\bullet_b(X;\mathbb{R}) := H^\bullet(C^\bullet_b(X;\mathbb{Z}), d).$$

Now as in the classical case one shows that $H^\bullet_b(G;\mathbb{Z}) \cong H^\bullet_b(BG;\mathbb{Z})$ and $H^\bullet_b(G;\mathbb{R}) \cong H^\bullet_b(BG;\mathbb{R})$. However, actually much more is true: Bounded cohomology does not see higher homotopy groups, whence:

Theorem 3.2 (Gromov-Brooks-Ivanov, cf. [23, 26])**.** *Let X be a space which has the homotopy type of a countable CW-complex. Then*

$$H^\bullet_b(X;\mathbb{Z}) \cong H^\bullet_b(\pi_1(X);\mathbb{Z}), \quad H^\bullet_b(X;\mathbb{R}) \cong H^\bullet_b(\pi_1(X);\mathbb{R}).$$

Finally, there is also a categorical interpretation of $H^\bullet_b(G;\mathbb{R})$, which was obtained by Bühler in his thesis [3]. Namely, for $H^\bullet_b(G;-)$ is the derived functor of $(-)^G$ in the (semi-abelian) category of Banach-G-modules. We will not go into this abstract definition any further; it basically means that, like usual cohomology, bounded cohomology can be computed from many different resolutions. One such resolution, the so-called boundary resolution of Burger and Monod, will be discussed below. To the best of our knowledge, there is no categorical approach to integral bounded group cohomology.

In the remainder of this section we present some standard material concerning bounded cohomology. Unless otherwise mentioned, this material can be found in [29, 3, 12].

3.2. **The comparison map I: Non-surjectivity for amenable groups.** The inclusion of subcomplexes $C^n_b(G;\mathbb{R}) \hookrightarrow C^n(G;\mathbb{R})$ induces, on the level of cohomology, a *comparison map*

$$c^\bullet = c^\bullet_G : H^\bullet_b(G;\mathbb{R}) \to H^\bullet_b(G;\mathbb{R}),$$

which *in general is neither injective nor surjective.*[1] Let us first explain, why it fails to be surjective in general. Digressing a little bit, let us first point out that the usual group cohomology $H^\bullet(F;\mathbb{R})$ vanishes for any finite group F. Indeed, by averaging

[1]As pointed out to me by M. Bucher, this half-sentence seems to be contained in every single paper on bounded cohomology.

over the finite group we obtain equivariant operators $A : C^n(G;\mathbb{R}) \to C^{n-1}(G;\mathbb{R})$, which satisfy $dA = \mathrm{Id}$. These then provide a contracting homotopy for the complex

$$\begin{tikzcd} 0 \arrow[r] & C^0(G;\mathbb{R})^G \arrow[r, "d"] & C^1(G;\mathbb{R})^G \arrow[r, "d"] \arrow[l, bend left, "A"] & C^2(G;\mathbb{R})^G \arrow[r, "d"] \arrow[l, bend left, "A"] & C^3(G;\mathbb{R})^G \arrow[r, "d"] \arrow[l, bend left, "A"] & \ldots, \arrow[l, bend left, "A"] \end{tikzcd}$$

showing that $H^n(F;\mathbb{R}) = \{0\}$ for $n > 0$. This argument works only for finite groups, since these are the only ones admitting an equivariant averaging operator. However, in bounded cohomology we can generalize the above argument quite a bit. Recall that a group G is called *amenable* if $l^\infty(G)$ admits a G-invariant *mean*, i.e. a linear fuctional $\mathfrak{m} \in l^\infty(G)'$ such that $\mathfrak{m}(1_G) = 1$, $\|\mathfrak{m}\| = 1$ and $f \geq 0 \Rightarrow \mathfrak{m}(f) \geq 0$. Using such a mean it is easy to construct averaging operators $A : C_b^n(G;\mathbb{R}) \to C_b^{n-1}(G;\mathbb{R})$ which provide a contracting homotopy

$$\begin{tikzcd} 0 \arrow[r] & C_b^0(G;\mathbb{R})^G \arrow[r, "d"] & C_b^1(G;\mathbb{R})^G \arrow[r, "d"] \arrow[l, bend left, "A"] & C_b^2(G;\mathbb{R})^G \arrow[r, "d"] \arrow[l, bend left, "A"] & C_b^3(G;\mathbb{R})^G \arrow[r, "d"] \arrow[l, bend left, "A"] & \ldots. \arrow[l, bend left, "A"] \end{tikzcd}$$

We thus deduce:

Proposition 3.3. *If G is amenable, then $H_b^n(G;\mathbb{R}) = \{0\}$ for all $n > 0$.*

We will not have the time to discuss amenable groups in detail; let us just remark that

- $\mathbb{Z}$ and more generally, abelian groups, are amenable by the Markov-Kakutani fixed point theorem;
- finite groups are (obviously) amenable;
- extensions and directed unions of amenable groups are amenable;
- combining these three examples, directed union of finite-by-solvable groups are amenable; these are called *elementary amenable*;
- finitely generated groups of subexponential growth are amenable.

It was a long standing open question, whether every amenable group is elementary amenable. The question was answered in the negative by constructing non-elementary amenable groups of subexponential (in fact, intermediate) growth. The first such examples were constructed by Grigorchuk in 1980.

After this short digression we now return to our comparison map. From the amenability of $\mathbb{Z}^n$ we deduce that the comparison map is not surjective in any degree > 0:

Corollary 3.4. *For $n > 0$ the comparison map*

$$c_{\mathbb{Z}}^n : \{0\} \cong H_b^n(\mathbb{Z}^n;\mathbb{R}) \to H^n(\mathbb{Z}^n;\mathbb{R}) \cong \mathbb{R}$$

is not surjective.

3.3. **The comparison map II: Non-injectivity and quasimorphisms.** More surprising than the non-surjectivity of the comparison map is probably the non-injectivity. To understand what is going on here, we have to unravel definitions. We are going to work in the inhomogeneous bar resolution, and we will focus on low degrees. In degrees 0 and 1 there is nothing to gain: As in Exercise 2.4 one shows that for any group G one has $H_b^0(G;\mathbb{R}) \cong \mathbb{R}$ and $H_b^1(G;\mathbb{R}) = \mathrm{Hom}_b(G;\mathbb{R})$, the space of bounded homomorphisms into $\mathbb{R}$. However, $\mathbb{R}$ does not have bounded subgroups, whence:

Lemma 3.5. *For any group G we have $H_b^1(G;\mathbb{R}) = \{0\}$.*

Thus in order to find comparison maps with non-trivial kernel we have to go at least to degree 2. Here the relevant part of the inhomogeneous bar resolution is given by

$$\begin{tikzcd}
\mathrm{Map}(G;\mathbb{R}) \arrow[r, "\partial^1"] & \mathrm{Map}(G^2;\mathbb{R}) \arrow[r, "\partial^2"] & \mathrm{Map}(G^3;\mathbb{R}) \\
l^\infty(G;\mathbb{R}) \arrow[r, "\partial^1"] \arrow[u] & l^\infty(G^2;\mathbb{R}) \arrow[r, "\partial^2"] \arrow[u] & l^\infty(G^3;\mathbb{R}) \arrow[u]
\end{tikzcd}$$

where $\partial^1 f(g,h) = f(h) - f(gh) + f(h)$. Now assume that

$$\alpha = [c] \in EH_b^2(G;\mathbb{R}) := \ker(c_G^2).$$

Since $c_G^2(\alpha) = 0$ there exists $f \in \mathrm{Map}(G;\mathbb{R})$ such that $\partial^1 f = c$. Note that

$$\sup_{g,h\in G} |f(gh) - f(g) - f(h)| = \sup_{g,h\in G} |c(g,h)| < \infty.$$

Thus f is almost a homomorphism. We introduce a name for such functions:

Definition 3.6. A function $f : G \to \mathbb{R}$ is called a *quasimorphism* of *defect* $D(f)$ if

$$D(f) := \sup_{g,h\in G} |f(gh) - f(g) - f(h)| < \infty.$$

Two quasimorphisms f_1, f_2 are called *equivalent* if $\|f_1 - f_2\|_\infty < \infty$. We denote by $\widetilde{\mathcal{Q}}(G)$ the space of quasimorphisms on G and by $\mathcal{Q}(G)$ the space of equivalence classes of quasimorphisms.

By the previous considerations we have a surjective linear map

$$q : \widetilde{\mathcal{Q}}(G) \to EH_b^2(G;\mathbb{R}), \quad f \mapsto [df].$$

Exercise 3.7. Show that the kernel of q is given by $l^\infty(G) \oplus \mathrm{Hom}(G;\mathbb{R})$.

As an immediate consequence of the exercise we have:

Proposition 3.8. *There map q induces an isomorphism*

$$\frac{\mathcal{Q}(G)}{\mathrm{Hom}(G;\mathbb{R})} = \frac{\widetilde{\mathcal{Q}}(G)}{l^\infty(G) \oplus \mathrm{Hom}(G;\mathbb{R})} \cong EH_b^2(G;\mathbb{R}).$$

Since it is slightly inconvenient to work with equivalence classes of quasimorphisms all the time, it is useful to find canonical representatives.

Proposition 3.9. *Let f be a quasimorphism.*

(i) *For every $g \in G$ the limit*

$$\widetilde{f}(g) := \lim_{n\to\infty} \frac{f(g^n)}{n}$$

exists.

(ii) *$\widetilde{f}$ is a quasimorphism, which is equivalent to f.*

(iii) *$\widetilde{f}$ is homogeneous, i.e. $f(g^n) = n \cdot f(g)$ for all $n \geq 0$.*

(iv) $\widetilde{f}$ *is the unique homogeneous quasimorphism equivalent to* f.

Exercise 3.10. Prove Proposition 3.9.

In view of the proposition, we can identify $\mathcal{Q}(G)$ with the space of *homogeneous* quasimorphisms on G. Thus:

Corollary 3.11. *Let* G *be a group. Then* c_G^2 *is injective if and only if every homogeneous quasimorphism on* G *is a homomorphism.*

Combining this with Proposition 3.3 we deduce:

Corollary 3.12. *If* G *is an amenable group, then every homogeneous quasimorphism on* G *is a homomorphism.*

The following special case is noteworthy:

Corollary 3.13. *Let* $f : G \to \mathbb{R}$ *be a homogeneous quasimorphism. Let* $a, b \in G$ *and assume that* a *and* b *commute. Then* $f(ab) = f(a) + f(b)$.

Proof. Let $H := \langle a, b\rangle$. Then H is abelian, hence amenable, and thus $f|_H$ is a homomorphism. □

The following lemma collects further properties of homogeneous quasimorphisms:

Lemma 3.14. *Let* $f : G \to \mathbb{R}$ *be a homogeneous quasimorphism. Then the following hold:*

(i) $f(g^{-1}) = -f(g)$.
(ii) $f(g^n) = n \cdot f(g)$ *for all* $n \in \mathbb{Z}$.
(iii) f *is conjugation-invariant.*
(iv) *If* $N \lhd G$ *is normal,* $p : G \to G/N$ *is the canonical projection and* $f|_N = 0$, *then there exists a homogeneous quasimorphism* $F : N \to \mathbb{R}$ *with* $f = F \circ p$.
(v) f *is a homomorphism if and only if* $f|_{[G,G]} = 0$.

Proof. (i) $0 = f(e) = f(g^n g^{-n}) = f(g^n) + f((g^{-1})^n) + o(1) = n \cdot f(g) + n \cdot f(g^{-1}) + o(1)$. Now divide by n and let $n \to \infty$. (ii) immediate from (i). (iii) Compute

$$n \cdot f(hgh^{-1}) = f(hg^n h^{-1}) = f(h) + f(g^n) + f(h^{-1}) + o(1) = n \cdot f(g) + o(1),$$

divide by n and let $n \to \infty$. (iv) For every coset gN pick a representative g and set $F_0(gN) := f(g)$. It is easy to check that F_0 is a quasimorphism. Let F be the homogenization of F_0; then one checks that $p^*F - f$ is a bounded homogeneous function, hence equal to 0. (v) If f is a homomorphism, then clearly $f|_{[G,G]} = 0$. Conversely, assume that $f|_{[G,G]} = 0$ and denote by $G_{\mathrm{ab}} := G/[G,G]$ the abelianization. By (iv), we then find a homogeneous quasimorphism $F : G_{\mathrm{ab}} \to \mathbb{R}$ with $f = F \circ p$, where $p : G \to G_{\mathrm{ab}}$ is the canonical projection. Now by Corollary 3.12, F is a homomorphism, hence f is a homomorphism as well. □

We now return to the problem of (non-) injectivity of the comparison map.

Example 3.15 (Brooks)**.** Let $G = F_2$ be the free group with basis $\{a, b\}$. Identify elements of G with reduced words over $\{a, b\}$. Given such a reduced word w, denote by $\#_{ab}(w)$ the number of occurrences of the sequence ab in w, and define $\#_{(ab)^{-1}}(w)$ similarly. Then

$$\varphi_{ab} : F_2 \to \mathbb{Z}, \quad w \mapsto \#_{ab}(w) - \#_{(ab)^{-1}}(w)$$

is a quasimorphism of defect 1.

Corollary 3.16. *$c^2_{F_2}$ is not injective.*

Proof. The homogenization $f := \widetilde{\varphi}_{ab}$ vanishes on a and b and satisfies $f(ab) = 1$. If $c^2_{F_2}$ was injective, then f was a homomorphism by Corollary 3.11. However, there is no homomorphism f of F_2 with $f(a) = f(b) = 0$ and $f(ab) = 1$. □

With (quite) a bit more work one can push this idea to show that, in fact [22]

$$\dim H^2_b(F_2; \mathbb{R}) = \dim EH^2_b(F_2; \mathbb{R}) = \infty.$$

The obvious idea is to construct counting quasimorphisms φ_w for arbitrary $w \in F_2$. What is not obvious, is that sufficiently many of these quasimorphisms are linearly independent. This was established by Grigorchuk in 1994, after several wrong proofs had appeared.

3.4. **Aside: A geometric disaster.** The most powerful tool in cohomology of topological spaces is arguably the excision theorem. It says that if $X = U \cup V$ is a space build from subspaces U and V, then $H^\bullet(X; \mathbb{R})$ can be computed from the cohomologies of U, V and $U \cap V$. For example, if X is the figure eight, then we can decompose it into two circles U and V, intersecting in a point. Thus the cohomology of the figure eight can be computed from the cohomology of the circle and the cohomology of a point. This is true for all generalized cohomology theories in the sense of Eilenberg-Steenrod, even for more exotic ones like K-theory. Now, for bounded cohomology this excision theorem fails in the most horrible way. For the circle S^1 we compute

$$H^n_b(S^1; \mathbb{R}) = H^n_b(\mathbb{Z}; \mathbb{R}) = \{0\} \quad (n \geq 1).$$

However, for the figure eight we have

$$H^n_b(8; \mathbb{R}) = H^n_b(F_2; \mathbb{R}),$$

which is not only non-zero, but in fact infinite-dimensional in degree 2. This is the reason why bounded cohomology of topological spaces has evaded almost any computational attempts beyond degree 2 .

3.5. **An interpretation of H^2_b.** We have seen that $H^2(G; \mathbb{R})$ classifies central $\mathbb{R}$-extensions of G. What does $H^2_b(G; \mathbb{R})$ classify? Since H^2 classifies central extensions and EH^2_b classifies quasimorphisms, it is reasonable to expect that H^2_b classifies a combination of the two. This is indeed the case, although it has not been spelled out explicitly in the literature. We suggest here the following new terminology;

Definition 3.17. A *quasi-corner* over G is a diagram

$$\begin{tikzcd} \widetilde{G} \arrow[r, "f"] \arrow[d, "p"] & \mathbb{R} \\ G & \end{tikzcd}$$

such that p is onto, $\ker(p)$ is central in $\widetilde{G}$ and f is a homogeneous quasimorphism.

We write $C(f,p)$ to denote the above quasi-corner. Note that every quasimorphism f on G defines a quasi-corner $C(f,\mathrm{id}_G)$. This allows us to consider quasimorphism as special quasi-corners. We now introduce a notion of equivalence of quasi-corners, which generalizes the notion of being cohomologous for quasimorphisms. For this let $C(f_1,p_1)$ and $C(f_2,p_2)$ be quasi-corners as above. Let $\widetilde{G}$ be the pushout

$$\begin{tikzcd} & \widetilde{G} \arrow[dl, "\pi_1"'] \arrow[dd, "p"] \arrow[dr, "\pi_2"] & \\ \widetilde{G_1} \arrow[dr, "p_1"'] & & \widetilde{G_2} \arrow[dl, "p_2"] \\ & G & \end{tikzcd}$$

where $p := p_1 \circ \pi_1 = p_2 \circ \pi_2$. Then p is surjective, and since the kernels of p_1 and p_2 are abelian, its kernel is amenable. In other words, $p : \widetilde{G} \to G$ is an (in general non-central) amenable extension.

Definition 3.18. The quasi-corners $C(f_1,p_1)$ and $C(f_2,p_2)$ are *equivalent* if

$$(\pi_1^* f_1 - \pi_2^* f_2)|_{[\widetilde{G},\widetilde{G}]} \equiv 0.$$

We denote by $\mathcal{QC}(G)$ the set of equivalence classes of quasi-corners over G.

Note that in view of Proposition 3.14 the above condition is equivalent to $\pi_1^* f_1 - \pi_2^* f_2$ being a homomorphism. We now relate equivalence classes of quasi-corners over G to classes in $H_b^2(G;\mathbb{R})$; for this the key observation as as follows:

Proposition 3.19 (Invariance under amenable extensions)**.** *For any amenable extension $p : \widetilde{G} \to G$ the map $p^* : H_b^2(G;\mathbb{R}) \to H_b^2(\widetilde{G};\mathbb{R})$ is an isomorphism. In particular, this is the case for any central extension.*

Proof. Let us denote by C the kernel of p so that we have a short exact sequence

$$\{e\} \to C \xrightarrow{i} \widetilde{G} \xrightarrow{p} G \to \{e\},$$

which we can exploit cohomologically. Indeed, a standard tool in general homological algebra is the five-term exact sequence of a derived functor. In the case of bounded cohomology this five-term exact sequence for the central extension ξ reads[2]

$$0 \to H_b^2(G;\mathbb{R}) \xrightarrow{p^*} H_b^2(\widetilde{G};\mathbb{R}) \to H_b^2(C;\mathbb{R}) \to H_b^3(G;\mathbb{R}) \to H_b^3(\widetilde{G};\mathbb{R}).$$

Now C is assumed amenable, hence the proposition follows from Proposition 3.3. $\square$

[2]If you know classical group cohomology, you might know a similar sequence starting from H^1; in bounded cohomology the degree is shifted by one, since H_b^1 is always trivial. The proof (essentially a spectral sequence argument) is the same.

Given a quasi-corner $C = C(f, p)$ we define

$$\alpha_b(C) := (p^*)^{-1}(df) \in H_b^2(G; \mathbb{R}).$$

We then say that C *realizes* the class $\alpha_b(C)$. We observe:

Proposition 3.20. *If two quasi-corners are equivalent, then they realize the same bounded cohomology class.*

Proof. This is immediate from Propositions 3.19, 3.8 and 3.9. □

We thus obtain a well-defined map

$$\alpha : \mathcal{QC}(G) \to H_b^2(G; \mathbb{R}), \quad [C] \mapsto \alpha_b(C).$$

Proposition 3.21. *The map $\alpha : \mathcal{QC}(G) \to H_b^2(G; \mathbb{R})$ is a bijection.*

Proof. In view of Proposition 3.14.(v), injectivity follows again from Propositions 3.19, 3.8 and 3.9. Indeed, two quasi-corners $C(f_1, p_1)$ and $C(f_2, p_2)$ represent the same class in bounded cohomology if and only if $[d\pi_1^* f_1] = [d\pi_2^* f_2]$, which by Proposition 3.14.(v) precisely means that $\pi_1^* f_1 - \pi_2^* f_2$ is a homomorphism. By definition this means that the two quasi-corners are equivalent. It remains to show that every bounded cohomology class $\alpha_b \in H_b^2(G; \mathbb{R})$ can be realized by a quasi-corner. For this let $\alpha := c_G^2(\alpha_b) \in H^2(G; \mathbb{R})$. By Corollary 2.8 there exists a central extension ξ of G, along which the pullback of α vanishes. (In general, ξ will be an $\mathbb{R}$-extension, but in many favorable situations e.g. if α is rational we can actually pick a $\mathbb{Z}$-extension.) We now fix one such central extension

$$\xi = (0 \to C \to \widetilde{G} \xrightarrow{p} G \to 1),$$

where C is some subgroup of $\mathbb{R}$ and $p^*\alpha = 0$. By naturality of the comparison map we now have

$$c_{\widetilde{G}}^2(p^*(\alpha_b)) = p^* c_{\widetilde{G}}^2(\alpha_b) = p^*\alpha = 0,$$

whence

$$p^*\alpha_b \in EH_{cb}^2(G; \mathbb{R}).$$

By Proposition 3.8 and Proposition 3.9 this implies that there exists a homogeneous quasimorphism $f : \widetilde{G} \to \mathbb{R}$ such that

$$df = p^*\alpha_b,$$

whence $\alpha_b = \alpha_b(C(f, p))$. □

The reason why the quasi-corner model for $H_b^2(G; \mathbb{R})$ is useful is the following naturality property, which we will use many times:

Exercise 3.22. Show that the obvious pullback of quasi-corners (descends to equivalence classes and) induces pullback on the level of bounded cohomology.

3.6. **The case of surface groups.** In the case of surface groups the situation is considerably simplified by the existence of a universal central extension $p_{\Gamma_g} : \widetilde{\Gamma_g} \to \Gamma_g$ as defined in Section 2.6. Indeed, assume that we are given two quasi-corners

$$\begin{tikzcd} \widetilde{G} \arrow[r, "f_G"] \arrow[d, "p_G"] & \mathbb{R} \\ G & \end{tikzcd} \qquad\qquad \begin{tikzcd} \widetilde{H} \arrow[r, "f_H"] \arrow[d, "p_H"] & \mathbb{R}, \\ H & \end{tikzcd}$$

with $\ker(p_G) \cong \ker(p_H) \cong \mathbb{Z}$, representing classes α_b^G and α_b^H respectively. Assume moreover that we are given homomorphisms $\rho_G : \Gamma \to G$ and $\rho_H : \Gamma \to H$. By Proposition 2.9 we then find lifts making the diagram

$$\begin{tikzcd} \mathbb{R} & \widetilde{G} \arrow[l, "f_G"'] \arrow[d, "p_G"'] & \widetilde{\Gamma_g} \arrow[l, "\widetilde{\rho_G}"'] \arrow[r, "\widetilde{\rho_H}"] \arrow[d, "p_{\Gamma_g}"'] & \widetilde{H} \arrow[r, "f_H"] \arrow[d, "p_H"] & \mathbb{R} \\ & G & \Gamma_g \arrow[l, "\rho_G"] \arrow[r, "\rho_H"'] & H & \end{tikzcd} \tag{3.1}$$

commute. Then we have the following special case of Proposition 3.21:

Corollary 3.23. *In the situation of* (3.1) *the following are equivalent:*

(i) $\rho_G^* \alpha_b^G = \rho_H^* \alpha_b^H \in H_b^2(\Gamma; \mathbb{R})$.
(ii) *The function* $f_G \circ \widetilde{\rho_G} - f_H \circ \widetilde{\rho_H} : \widetilde{\Gamma_g} \to \mathbb{R}$ *is a homomorphism.*
(iii) $(f_G \circ \widetilde{\rho_G} - f_H \circ \widetilde{\rho_H})|_{[\widetilde{\Gamma_g}, \widetilde{\Gamma_g}]} \equiv 0$.

4. Bounded Euler class and applications to surface group representations

Having developed the most elementary parts of the theory of bounded cohomology we now turn to a first application. We introduce the bounded Euler class and prove a special case of a famous theorem of Ghys [19, 20].

4.1. **The bounded Euler class and the translation number.** At this point we have assembled enough information about bounded cohomology to give a first application. For this we return to the situation of Section 2.7 and consider the central extension $p : \widetilde{G} \to G$ given by $G := \mathrm{Homeo}^+(S^1)$, $\widetilde{G} := \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$. Recall that this central extension corresponds to a class $e(S^1) \in H^2(G; \mathbb{Z})$, which is represented by the cocycle

$$c_\sigma(g, h) = (\sigma(g)\sigma(gh)^{-1}\sigma(g))(0).$$

A specific section $\sigma : G \to \widetilde{G}$ can be given as follows: Let $\sigma(f) \in \widetilde{G}$ be the unique lift of f with $\sigma(f)(0) \in [0, 1)$. Then, obviously, c_σ is bounded, hence defines a class

$$e_b(S^1) := [c_\sigma(g, h)] \in H_b^2(G; \mathbb{Z}).$$

Note that by definition, $e_b(S^1)$ is mapped to $e(S^1)$ under the comparison map, hence it is called a *bounded Euler class*. We also denote by $e_b^{\mathbb{R}}(S^1)$ the corresponding class in $H_b^2(G; \mathbb{R})$. We will usually shorten the notation to $e, e_b, e^{\mathbb{R}}, e_b^{\mathbb{R}}$, dropping the S^1 from the notation.

Now the *real bounded Euler class* $e_b^{\mathbb{R}}(S^1)$ can be represented by a quasi-corner of the form

$$\begin{tikzcd} \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) \arrow[r, "T"] \arrow[d, "p"] & \mathbb{R} \\ \mathrm{Homeo}^+(S^1), & \end{tikzcd}$$

and since $\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$ is perfect, the homogeneous quasimorphism $T : \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) \to \mathbb{R}$ is actually uniquely determined.

Proposition 4.1. (i) *For every $x \in \mathbb{R}$ the map $T_x : \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) \to \mathbb{R}$ given by*

$$T_x(f) = f(x) - x$$

is a quasimorphism.

(ii) *The quasimorphisms T_x are at mutually bounded distance, hence have a common homogenization $T : \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) \to \mathbb{R}$ given by*

$$Tf = \lim_{n\to\infty} \frac{f^n(x) - x}{n},$$

which is independent of $x \in \mathbb{R}$.

(iii) *If $p : \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) \to \mathrm{Homeo}^+(S^1)$ is the canonical projection, then $e_b^{\mathbb{R}}(S^1)$ is represented by the quasi-corner $C(-T, p)$.*

The quasimorphism T was discovered by Poincaré in 1889. It is called the *translation number quasimorphism* and plays a major role in the theory of dynamical systems.

Proof of Proposition 4.1. (i), (ii) The key observation is that for $f, g \in \widetilde{G}$ and $x \in \mathbb{R}$ we have

$$|T_x(f) - T_y(f)| = |(f(x) - x) - (f(y) - y)| \quad \leq \quad 1.$$

Indeed, (ii) is immediate from this observation, and (i) also follows since

$$T_x(f \circ g) - T_x(f) - T_x(g) = T_{g(x)}(f) - T_x(f).$$

It remains to show (iii). We will actually show the following more precise statement: If $\sigma : G \to \widetilde{G}$ is the unique section with $\sigma(f)(0) \in [0, 1)$ and e_σ is the corresponding cocycle representative of the Euler class of the extension p, then $p^*e_\sigma = dA$, where $A(f) = \lfloor f(0) \rfloor$. Since $|A(f) - T_0(f)| < 1$, this implies the statement. The key step in the proof is to show that for $f_1, f_2 \in \widetilde{G}$ we have

$$\lfloor (f_1f_2)(0) \rfloor = -\lfloor (f_1f_2)^{-1}(0) \rfloor. \tag{4.1}$$

Indeed, every $f \in \widetilde{G}$ satisfies

$$\lfloor f^{-1}\lfloor f(0) \rfloor \rfloor \leq \lfloor f^{-1}(f(0)) \rfloor = 0, \quad \lfloor f^{-1}\lfloor f(0) \rfloor \rfloor > \lfloor f^{-1}(f(0) - 1) \rfloor = -1,$$

leading to $\lfloor f^{-1}\lfloor f(0) \rfloor \rfloor = 0$. This now yields

$$0 = \lfloor f^{-1}(\lfloor f(0) \rfloor) \rfloor = \lfloor f^{-1}(0 + \lfloor f(0) \rfloor) \rfloor = \lfloor f^{-1}(0) + \lfloor f(0) \rfloor \rfloor = \lfloor f^{-1}(0) \rfloor + \lfloor f(0) \rfloor,$$

which for $f := f_1f_2$ yields (4.1). Thus if we denote $\overline{f} := f - A(f)$, then

$$\begin{aligned} (p^*e_\sigma)(f_1, f_2) \quad &= \quad i^{-1}(\sigma(p(f_1)p(f_2))^{-1}\sigma(p(f_1))\sigma(p(f_2))) \\ &= \quad (\overline{(f_1f_2)^{-1}}\bar{f}_1\bar{f}_2)(0) \end{aligned}$$

$$
\begin{aligned}
&= \quad (\overline{(f_1 f_2)^{-1}}\bar{f}_1(f_2 - \lfloor f_2(0) \rfloor)(0) \\
&= \quad (\overline{(f_1 f_2)^{-1}}(f_1 f_2 - \lfloor f_2(0) \rfloor - \lfloor f_1(0) \rfloor)(0) \\
&= \quad 0 - \lfloor f_2(0) \rfloor - \lfloor f_1(0) \rfloor - \lfloor (f_1 f_2)^{-1}(0) \rfloor \\
&\overset{(4.1)}{=} \quad \lfloor (f_1 f_2)(0) \rfloor - \lfloor f_2(0) \rfloor - \lfloor f_1(0) \rfloor) \\
&= \quad -dA(f_1, f_2).
\end{aligned}
$$

□

4.2. **Vanishing of bounded Euler classes.** Let G be a group and $\rho : G \to \mathrm{Homeo}^+(S^1)$ be a circle action. We have seen that $\rho^* e = 0$ if and only if the action lifts to the real line. What is the meaning of $\rho^* e_b = 0$? This condition of course implies that the action lifts, however, since the comparison map of G may have a huge kernel (as we have seen in the case of free groups), there is much more information to be gained from e_b.

Proposition 4.2. *Let $\rho : G \to \mathrm{Homeo}^+(S^1)$ be a circle action with $\rho^* e_b = 0$. Then $\rho(G)$ fixes a point on S^1.*

This is a baby version of a famous theorem of Ghys [19, 20], which says that $\rho^* e_b$ is a complete quasi-conjugacy invariant of circle actions. The short and elementary proof given below is due to M. Bucher and the author. The general theorem is based on a similar idea, but much more technical.

Proof. Let $u : G \to \mathbb{Z}$ be a bounded function with $\rho^* e_\sigma = du$, where σ is the special section defined above. By Proposition 2.7 we have a homomorphism

$$\widetilde{\rho} : G \to \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}), \quad \widetilde{\rho}(g) = \sigma(\rho(g)) \cdot i(u(g)).$$

In particular,

$$\widetilde{\rho}(g)(0) = \sigma(\rho(g))(0) + u(g).$$

Now, by choice of our section, $\sigma(\rho(g))(0) \in [0, 1)$, hence $\widetilde{\rho}(g)(0)$ is bounded. Set

$$F^+(\widetilde{\rho}) := \sup_{g \in G} \widetilde{\rho}(g)(0);$$

then $F^+(\widetilde{\rho})$ is a fixed point for $\widetilde{\rho}(G)$, hence its image in S^1 is a fixed point for $\rho(G)$. □

We will also need a version of the proposition for the *real* bounded Euler class:

Corollary 4.3. *Let $\rho : G \to \mathrm{Homeo}^+(S^1)$ be a circle action with $\rho^* e_b = 0$. Then $\rho([G, G])$ fixes a point on S^1.*

Proof. Consider the following segment of the Gersten sequence for G:

$$H^1(G, \mathbb{R}/\mathbb{Z}) = \mathrm{Hom}(G, \mathbb{R}/\mathbb{Z}) \xrightarrow{\delta} H^2_b(G, \mathbb{Z}) \longrightarrow H^2_b(G, \mathbb{R}) .$$

Then our assumption $\rho^*(e_b^{\mathbb{R}}) = 0$ implies that there exists a homomorphism $\chi : G \to \mathbb{R}/\mathbb{Z}$ such that $\rho^*(e_b) = \delta(\chi)$. We thus have to understand the connecting homomorphism δ. Observe that there is a canonical embedding $\iota : \mathbb{R}/\mathbb{Z} \to \mathrm{Homeo}^+(S^1)$ via the action by rotations, so we obtain a homomorphism $\widetilde{\chi} := \iota \circ \chi : G \to$

$\mathrm{Homeo}^+(S^1)$. Then a really unpleasant diagram chase shows that $\delta(\chi) = \widetilde{\chi}^* e_b$. We thus find

$$\rho^*(e_b) = \chi^*(e_b).$$

Now since $\mathbb{R}/\mathbb{Z}$ is abelian, the homorphism χ (and hence $\widetilde{\chi}$) vanishes on $[G, G]$. It follows that $\rho|_{[G,G]}^* e_b = 0$, whence $[G, G]$ has a fixed point. $\square$

4.3. **Surface groups, hyperbolizations and bounded fundamental class.** Let Σ_g be a closed oriented surface of genus g and $\Gamma_g := \pi_1(\Sigma_g)$. A representation $h : \Gamma_g \to \mathrm{PU}(1,1)$ will be called a *hyperbolization* if it is the holonomy representation of an oriented hyperbolic structure on Σ_g, and an anti-hyperbolization if it is the holonomy representation of a negatively-oriented hyperbolic structure on Σ_g. Here we think of $\mathrm{PU}(1,1)$ as the identity component of the automorphism group of the Poincaré disc $\mathbb{D}$. By classical results of Dehn, Nielsen and Baer, every discrete and faithful representation is either a hyperbolization or an anti-hyperbolization. Moreover, the unique outer automorphism of $\mathrm{PU}(1,1)$ swaps hyperbolizations and anti-hyperbolization. In studying discrete, faithful representations into $\mathrm{PU}(1,1)$ we can thus focus on hyperbolizations.

The action of $\mathrm{PU}(1,1)$ on $\mathbb{D}$ by fractional linear transformations extends continuously to the boundary $S^1 = \partial\mathbb{D}$. This boundary action gives rise to an embedding $\iota : \mathrm{PU}(1,1) \to \mathrm{Homeo}^+(S^1)$. Therefore, every hyperbolization $h : \Gamma_g \to \mathrm{PU}(1,1)$ gives rise to an embedding

$$\check{h} := \iota \circ h : \Gamma_g \to \mathrm{Homeo}^+(S^1),$$

which for lack of a better name we call a *hyperbolic boundary action*. The following is a weak version of hyperbolic stability:

Lemma 4.4. *Any two hyperbolic boundary actions $\check{h}_1, \check{h}_2 : \Gamma_g \to \mathrm{Homeo}^+(S^1)$ are conjugate.*

The strong version of hyperbolic stability says that they are actually conjugate by a Hölder continuous homomorphism, but we will not need this fact here. We recall that the automorphism group $\mathrm{Aut}(G)$ of a group G acts on the (bounded) cohomology ring by functoriality. It is a classical fact in group cohomology, that the action of the inner automorphism group on cohomology is trivial, and this fact (and its proof) carry over to bounded cohomology. We deduce:

Corollary 4.5. *Let $\alpha \in H^\bullet(\mathrm{Homeo}^+(S^1), \mathbb{R})$ or $\alpha \in H_b^\bullet(\mathrm{Homeo}^+(S^1), \mathbb{R})$ and let $\check{h} : \Gamma_g \to \mathrm{Homeo}^+(S^1)$ be a hyperbolic boundary action. Then the class $\alpha|_{\Gamma_g} := \check{h}^*\alpha$ is independent of the choice of hyperbolic boundary action.*

This allows us to define:

Definition 4.6. The *fundamental class*, respectively *bounded fundamental class* of Σ_g are defined as $\kappa^{\Sigma_g} := e^{\mathbb{R}}(S_1)|_{\Gamma_g} \in H^2(\Gamma_g)$, respectively $\kappa_b^{\Sigma_g} := e_b^{\mathbb{R}}(S_1)|_{\Gamma_g} \in H_b^2(\Gamma_g)$.

The name fundamental class will be justified below, when we relate it to the hyperbolic volume form. For the moment, it is just a name. The term bounded fundamental class was introduced in [8], where the importance of this class was first fully realized. Namely, it is indeed fundamental in the sense that it detects certain types of small subgroups:

Definition 4.7. A subgroup $H < \mathrm{PU}(1,1)$ is called *elementary* if it has a finite orbit in $\overline{\mathbb{D}}$.

Theorem 4.8 (Detecting elementary subgroups with bounded cohomology)**.** *Let $\Lambda < \Gamma_g$ be a subgroup of a surface group. Assume that $(\kappa_b^{\Sigma_g})|_\Lambda = 0 \in H_b^2(\Lambda;\mathbb{Z})$. Then the image of Λ under any hyperbolization is elementary. If Λ is normal in Γ_g, then Λ is trivial.*

Proof. Let us fix a hyperbolization $h : \Gamma \to \mathrm{PU}(1,1)$ and an associated boundary action $\widetilde{h} : \Gamma \to \mathrm{Homeo}^+(S^1)$. Denote by $\mathcal{F}$ the set of $\widetilde{h}([\Lambda,\Lambda])$-fixed points in S^1. Since $(\widetilde{h}|_\Lambda)^* e_b^{\mathbb{R}} = 0$ we deduce from Corollary 4.3 that $\mathcal{F}$ is non-empty. By construction, $\mathcal{F}$ is $\widetilde{h}(\Lambda)$-invariant, hence if $\mathcal{F}$ is finite then $h(\Lambda)$ is elementary. We may thus assume that $\mathcal{F}$ is infinite. However, the only subgroup of $\mathrm{PU}(1,1)$ with infinitely many fixed points on the circle is the trivial group. It follows that $\widetilde{h}([\Lambda,\Lambda]) = \{e\}$, whence h factors through the abelianization of Λ. Now every abelian subgroup of $\mathrm{PU}(1,1)$ is elementary. This proves the first statement, and the second statement follows from the fact that the hyperbolization of a surface group has no normal elementary subgroups. □

Before we turn to applications we need one more technical tool:

Lemma 4.9. *Let $\check{h} : \Gamma_g \to \mathrm{Homeo}^+(S^1)$ be a boundary hyperbolization and $\widetilde{\check{h}} : \widetilde{\Gamma_g} \to \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$ be a lift. Then*

$$T(\widetilde{\check{h}}(\widetilde{\Gamma_g})) \subset \mathbb{Z}.$$

Proof. Every $g \in \check{h}(\Gamma_g)$ is hyperbolic, hence has a fixed point on S^1. Thus, in every fiber of $\widetilde{\check{h}}(\widetilde{\Gamma_g})$ over $\check{h}(\Gamma_g)$ there is an element g^* which fixes a point in $\mathbb{R}$. For this elements we have $T(g^*) = 0$ by the explicit formula for T. Now if h is a different element in the same fiber, then $h = g^*\tau$, where τ is an integer translation. Now translations are central, hence $T(h) = T(g^*) + T(\tau) = T(\tau) \in \mathbb{Z}$ by Corollary 3.13. □

4.4. **A cohomological criterion for injectivity.** We now present a first application of bounded cohomology to representation theory of surface groups, based on Theorem 4.8. Concerning surface groups, we use the notation introduced in the previous section

Theorem 4.10 (Injectivity criterion)**.** *Let G be a group and $\alpha_b \in H_b^2(G;\mathbb{R})$. Assume that α_b admits a rational representative. If $\rho : \Gamma_g \to G$ is a representation with*

$$\rho^*\alpha_b = \lambda \cdot \kappa_b^{\Sigma_g}$$

for some $\lambda \neq 0$, then ρ is injective.

Proof. We represent α_b by a quasi-corner (f_G, p_G). Clearing denominators, we may assume without loss of generality that the class α_b actually has an integral representative. Then we can choose p_G so that $\ker(p_G) \cong \mathbb{Z}$. We also fix a hyperbolization

h and a corresponding boundary action $\check{h}$. Then both ρ and $\check{h}$ lift, giving rise to the diagram

$$\begin{tikzcd} \mathbb{R} & \widetilde{G} \arrow[l, "f_G"'] \arrow[d, "p_G"'] & \widetilde{\Gamma_g} \arrow[l, "\widetilde{\rho}"'] \arrow[r, "\widetilde{\check{h}}"] \arrow[d, "p_{\Gamma_g}"'] & \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) \arrow[r, "T"] \arrow[d, "p"] & \mathbb{R} \\ & G & \Gamma_g \arrow[l, "\rho"] \arrow[r, "\check{h}"'] & \mathrm{Homeo}^+(S^1). & \end{tikzcd} \tag{4.2}$$

In view of Corollary 3.23 our assumption thus amount to the existence of a homomorphism $\psi : \widetilde{\Gamma_g} \to \mathbb{R}$ such that

$$f_G \circ \widetilde{\rho} = -\lambda \cdot (T \circ \widetilde{\check{h}}) + \psi. \tag{4.3}$$

Now consider the group $\Lambda := \ker(\rho)$ and let $\widetilde{\Lambda} := p_{\Gamma_g}^{-1}(\Lambda)$. We have

$$p_G(\widetilde{\rho}(\widetilde{\Lambda})) = \rho(p_{\Gamma_g}(\widetilde{\Lambda})) = \rho(\Lambda) = \{0\} \Rightarrow \widetilde{\rho}(\widetilde{\Lambda}) \subset \mathbb{Z}.$$

In particular, $\widetilde{\rho}(\Lambda)$ is abelian, and thus $(f_G \circ \widetilde{\rho})|_\Lambda$ is a homomorphism. It then follows from (4.3) that $T \circ \widetilde{\check{h}}|_{\widetilde{\Lambda}}$ is a homomorphism. This implies that

$$\kappa_b^{\Sigma_g}|_\Lambda = \check{h}^* e_b^{\mathbb{R}}(S^1) = p_{\Gamma_g}^{-1}[d(T \circ \widetilde{\check{h}}|_{\widetilde{\Lambda}})] = 0.$$

We thus deduce from Theorem 4.8 that $\Lambda = \{e\}$, whence ρ is injective. □

4.5. **A cohomological criterion for discreteness.** A variation of the same ideas as in the proof of Theorem 4.10 can also be used to detect whether a representation $\rho : \Gamma_g \to G$ of a surface group into Lie group G has discrete image.[3]

In order to discuss discreteness of a representation, we clearly need to take the topology of G into account. The right way to do this is to develop a topological version of bounded cohomology; we will discuss this approach in some details below. However, for the present purpose we will choose an ad hoc approach which serves it purpose. Given a topological group G, let us call a quasi-corner

$$\begin{tikzcd} \widetilde{G} \arrow[r, "f"] \arrow[d, "p"] & \mathbb{R} \\ G & \end{tikzcd}$$

over G a *topological quasi-corner* is $\widetilde{G}$ is a topological group, p is a covering map and f is a continuous[4] homogeneous quasimorphism. Let us call a class $\alpha \in H_b^2(G;\mathbb{R})$ *continuous* if it can be represented by a topological quasi-corner. Then we have:

Theorem 4.11 (Discreteness criterion)**.** *Let G be a Lie group and $\alpha_b \in H_b^2(G;\mathbb{R})$ be continuous. Assume that α_b admits a rational representative. If $\rho : \Gamma_g \to G$ is a representation with*

$$\rho^*\alpha_b = \lambda \cdot \kappa_b^{\Sigma_g}$$

[3]Since the proof of the main theorem is rather technical and its ideas and techniques are not used in the sequel, this section can be skipped without loss of continuity. However, we will use the results later on.

[4]If G is assumed locally compact second countable then every Borel homogeneous quasimorphism is continuous, so the assumption can be weakened

for some $\lambda \in \mathbb{Q}^\times$*, then (it is injective and)* $\rho(\Gamma_g)$ *is discrete.*

Proof. Again we may after clearing denominators assume that α_b is integral and represented by a continuous quasi-corner $C(f_G, p_G)$ with $\ker(p) \cong \mathbb{Z}$. We may also assume that $f_G(\ker(p_G)) = \frac{1}{D} \cdot \mathbb{Z}$, where D is the denominator we cleared. As in the proof of Theorem 4.10 we then construct the diagram (4.2) and obtain the formula (4.3). Now define

$$L := \overline{\rho([\Gamma_g, \Gamma_g])}, \quad \widetilde{L} := p_G^{-1}(L).$$

Since p_G is a covering we have

$$\widetilde{L} := p_G^{-1}(\overline{\rho[\Gamma_g, \Gamma_g]}) = \overline{p_G^{-1}(\rho([\Gamma_g, \Gamma_g]))} = \overline{\widetilde{\rho}([\widetilde{\Gamma_g}, \widetilde{\Gamma_g}]) \cdot \ker(p_G)}.$$

We claim that $f_G(\widetilde{L}) \subset \mathbb{R}$ is discrete. Since f_G it is continuous, it suffices to show that in fact $f_G(\widetilde{\rho}([\widetilde{\Gamma_g}, \widetilde{\Gamma_g}]) \cdot \ker(p_G))$ is discrete. Now, since $\widetilde{\rho}([\widetilde{\Gamma_g}, \widetilde{\Gamma_g}]$ and $\ker(p_G)$ commute, it follows from Corollary 3.13 that

$$f_G(\widetilde{\rho}([\widetilde{\Gamma_g}, \widetilde{\Gamma_g}]) \cdot \ker(p_G)) = f_G(\widetilde{\rho}([\widetilde{\Gamma_g}, \widetilde{\Gamma_g}])) + f_G(\ker(p_G)).$$

Concerning the first summand, the identity (4.3) in combination with Lemma 4.9 yields

$$f_G(\widetilde{\rho}([\widetilde{\Gamma_g}, \widetilde{\Gamma_g}])) = -\lambda \cdot (T \circ \widetilde{\widetilde{h}})([\widetilde{\Gamma_g}, \widetilde{\Gamma_g}]) \subset \lambda \cdot \mathbb{Z},$$

since the homomorphism ψ vanishes on commutators. Since the second summand is contained by $\frac{1}{D}\mathbb{Z}$ by assumption and since λ was assumed rational, we deduce that $f_G(\widetilde{L}) \subset \mathbb{R}$ is indeed discrete.

Let now $\widetilde{L}^o$ denote the identity component of $\widetilde{L}$. Since $\widetilde{L}^o$ is connected and $f_G(\widetilde{L})$ is discrete we have $f_G(\widetilde{L}^o) \equiv 0$. Combining this with the observation that p_G is an open map and hence $p_G(\widetilde{L}^o) = L^o$ we deduce that

$$0 = [df_G|_{\widetilde{L}^o}] = p_G^* \alpha_b|_{\widetilde{L}^o} = (p_G|_{\widetilde{L}^o})^*(\alpha_b|_{L^o}),$$

hence $\alpha_b|_{L^o} = 0$.

Now consider $\Delta := \rho^{-1}(\rho[\Gamma_g, \Gamma_g] \cap L^o)$. We have

$$(\rho|_\Delta)^*(\alpha_b) = (\rho|_\Delta)^*(\alpha_b|_{L^o}) = 0,$$

hence $(\lambda \cdot \kappa_b^{\Sigma_g})|_\Delta = 0$ by assumption. Since $\lambda \neq 0$ we deduce that $\kappa_b^{\Sigma_g}|_\Delta = 0$ and thus $\Delta = 0$ by Theorem 4.8. Consequently, $\rho[\Gamma_g, \Gamma_g] \cap L^o = \{e\}$.

Up to now, everything works for arbitrary topological groups G. Now we use that G is a Lie group. Namely, the closed subgroup $L < G$ is a Lie group, whence L^o is open in L. It follows that $\rho([\Gamma_g, \Gamma_g]) \cap L^o$ is dense in L^o, hence $L^o = \{e\}$. It follows that L, and consequently $\rho([\Gamma_g, \Gamma_g])$ is discrete.

Now $\rho(\Gamma)$ normalizes $\rho([\Gamma_g, \Gamma_g])$, hence $\overline{\rho(\Gamma)}$ normalizes $\overline{\rho([\Gamma_g, \Gamma_g])} = \rho([\Gamma_g, \Gamma_g])$. Consequently, $\rho([\Gamma_g, \Gamma_g])$ centralizes the identity component $\overline{\rho(\Gamma_g)}^o$. Now $\rho(\Gamma_g)$ is dense in $\overline{\rho(\Gamma_g)}$ and $\overline{\rho(\Gamma_g)}^o$ is open. Thus, if we assume $\overline{\rho(\Gamma_g)}^o \neq \{e\}$, then we can find $\gamma \in \Gamma_g$ with $\rho(\gamma) \in \overline{\rho(\Gamma_g)}^o \setminus \{e\}$. We then deduce that $\rho(\gamma)$ centralizes $\rho([\Gamma_g, \Gamma_g])$.

Now apply Theorem 4.10 and deduce that ρ is injectivity. It follows that γ centralizes $[\Gamma_g, \Gamma_g]$. However, the centralizer of $[\Gamma_g, \Gamma_g]$ in Γ_g is trivial. This contradiction show that $\overline{\rho(\Gamma_g)}^o = \{e\}$, whence $\rho(\Gamma_g)$ is discrete. □

In the proof we clearly use that $\lambda \in \mathbb{Q}^\times$ and that G is a Lie group. It is unclear to us, whether these assumptions are necessary. Rationality of λ is used in an essential way, whereas the Lie group property of G seems less essential (and is unfortunately very restrictive in terms of applications). In any case, at least for representations of surface groups into Lie groups we have found a cohomologival criterion which guarantees that the representation in question is discrete and faithful.

On first sight our criterion might not look very useful, since it basically replaces a mysterious condition by an even more mysterious one. The point we will have to make below is that in many interesting geometric situations one can actually check the cohomological condition. This will in particular be the case for maximal and Shilov-Anosov representations. Before we can make the link, however, we need to study some more bounded cohomology.

5. Bounded cohomology II: Bounded Kähler classes and boundary resolutions

5.1. **Continuous bounded cohomology of Lie groups.** There is a version of bounded cohomology for topological groups, due to Burger and Monod [9, 10, 11, 29]. Given a topological group G one can replace in the definition of bounded cohomology the space $l^\infty(G^{n+1};\mathbb{R})$ of bounded functions by the space of *continuous bounded functions* $C_b(G^{n+1};\mathbb{R})$. The resulting cohomology is called the *continuous bounded cohomology* of G and denoted $H^\bullet_{cb}(G;\mathbb{R})$. As for bounded cohomology one shows $H^0_{cb}(G;\mathbb{R}) \cong \mathbb{R}$ and $H^1_{cb}(G;\mathbb{R}) = \{0\}$. Thus the first interesting continuous bounded cohomology is the second one.

What is the second continuous bounded cohomology of a Lie group? We first recall that every Lie group G is of a semidirect product $G = R_uG \rtimes G_{ss}$, where R_uG is solvable and G_{ss} is semisimple with trivial center, hence decomposes as a product of simple factors. Now G_{ss} decomposes further as $G_{ss} = G^c_{ss} \times G^{nc}_{ss}$, where G^c_{ss} is compact and all simple factors of G^{nc}_{ss} are non-compact.

Proposition 5.1. $H^2_{cb}(G;\mathbb{R}) = H^2_{cb}(G^{nc}_{ss})$.

Proof. The 5-term sequence which we used in the proof of Proposition 3.19 exists also in continuous bounded cohomology. Thus, taking the quotient of a group by an amenable subgroup does not change the second bounded cohomology. Now R_uG is solvable and G^c_{ss} is compact, hence both are amenable. □

The result holds actually in arbitrary degrees and can be proved directly by using averaging operators constructing the means of R_uG and G^c_{ss}. We also mention the following result of Monod, which is special to degree 2, and can also be deduced from the five-term sequence:

Lemma 5.2. $H^2_{cb}(G_1 \times G_2;\mathbb{R}) \cong H^2_{cb}(G_1;\mathbb{R}) \oplus H^2_{cb}(G_2;\mathbb{R})$.

Combining this with the proposition we see that in our study of $H^2_{cb}(G;\mathbb{R})$ we can restrict attention to non-compact simple Lie groups. As in the case of bounded cohomology we have a comparison map $H^2_{cb}(G;\mathbb{R}) \to H^2_c(G;\mathbb{R})$, where the right hand side denotes cohomology with continuous cochains. Now we have:

Proposition 5.3. *For every semisimple non-compact Lie group G the degree 2 comparison map $H^2_{cb}(G;\mathbb{R}) \to H^2_c(G;\mathbb{R})$ is injective.*

Proof. Using Monod's lemma one immediately reduces to the simple case. As in the case of usual bounded cohomology one shows that the kernel of the comparison map is given by by continuous homogeneous quasimorphisms modulo homomorphisms. Thus the proposition amounts to showing that every continuous homogeneous quasimorphism on a simple Lie group is trivial. This follows from a general property of simple Lie groups called *bounded generation by unipotents.* We give the details for $G = \mathrm{SL}_n(\mathbb{R})$. Let $g_{ij}(t) := \exp(tE_{ij})$, where E_{ij} is the elementary matrix which is 1 at the ij-th entry and 0 everywhere else. Then for $i \neq j$

$$\lim_{s\to\infty} g_{ii}(s)g_{ij}(t)g_{ii}(s)^{-1} = e,$$

hence for every continuous homogeneous quasimorphism f and $i \neq j$ we have by Proposition 3.14

$$f(g_{ij}(t)) = \lim_{s\to\infty} f(g_{ij}(t)) = \lim_{s\to\infty} f(g_{ii}(s)g_{ij}(t)g_{ii}(s)^{-1}) = 0.$$

Now, by the Gauss algorithm, every matrix in $SL_n(\mathbb{R})$ can be written as a product of no more than $100n^2$ matrixes of the form $g_{ij}(t)$ with $i \neq j$. Therefore, f is bounded by $100n^2 \cdot D(f)$. However, a bounded homogeneous quasimorphism is trivial. □

So in order to understand $H^2_{cb}(G;\mathbb{R})$ for Lie groups, it suffices to understand $H^2_c(G;\mathbb{R})$ for simple Lie groups.

5.2. **Bounded Kähler class and Hermitian Lie groups.**

Example 5.4. Consider the action of $G := \mathrm{PU}(1,1)$ on the Poincaré disc $\mathbb{D}$ and denote by $\omega \in \Omega^2(\mathbb{D})^G$ the hyperbolic volume form $\omega = \frac{1}{1-(x^2+y^2)^2}dxdy$. Given $(x_0, x_1, x_2) \in \mathbb{D}$ denote by $\Delta(x_0, x_1, x_2)$ the geodesic triangle with corners x_0, x_1, x_2 and fix a basepoint $o \in \mathbb{D}$. Then we obtain a homogeneous 2-cocycle $c_{\omega,o} : G^3 \to \mathbb{R}$ given by

$$c_{\omega,o}(g_0, g_1, g_2) := \frac{1}{2\pi}\int_{\Delta(g_0.o,g_1.o,g_2.o)} \omega.$$

Exercise 5.5. Show that the cohomology class $\kappa_G := [c_{\omega,o}] \in H^2_c(G,\mathbb{R})$ is independent of the choice of basepoint o. [Hint: Use that ω is closed.]

We refer to κ_G as the *Kähler class* of G. It turns out that

$$H^2_c(G,\mathbb{R}) \cong \mathbb{R}\cdot\kappa_G.$$

Since every geodesic triangle is contained in an ideal triangle (of volume π) the cocycle $c_{\omega,o}$ is bounded by 1/2. It thus defined a class $\kappa^b_G := [c_{\omega,o}] \in H^2_c(G,\mathbb{R})$, called the *bounded Käbler class.* By injectivity of the comparison map, it is the unique pre-image of the Kähler class in bounded cohomology.

Example 5.4 is the prototype of a very general theory based on the following observation of Cartan:

Lemma 5.6 (E. Cartan)**.** *Let G be a simple Lie group, $\mathcal{X} = G/K$ the associated symmetric space and $\omega \in \Omega^n(\mathcal{X})^G$ a G-invariant form. Then ω is closed.*

Proof. Let $o \in \mathcal{X}$ be a basepoint and assume $\omega \in \Omega^n(\mathcal{X})^G$. Since the geodesic reflection σ_o at o normalizes G we have $\sigma_o^*\omega \in \Omega^n(\mathcal{X})^G$. Now, $\sigma_o^*\omega = (-1)^n\omega$; indeed, the relation holds at the basepoint o and both sides are G-invariant. We can apply the same argument also to $d\omega$ to obtain $\sigma_o^*(d\omega) = (-1)^{n+1}d\omega$. Now we get

$$(-1)^n d\omega = d((-1)^n\omega) = d\sigma_o^*\omega = \sigma_o^*(d\omega) = (-1)^{n+1}d\omega,$$

showing that $d\omega = 0$. □

By Cartan's lemma, we can integrate invariant forms on $\mathcal{X}$ over geodesic simplices and thereby construct a map

$$I : \Omega^n(\mathcal{X})^G \to H^n_c(G;\mathbb{R}), \quad I(\omega) = \left[(g_0,\dots,g_n) \mapsto \frac{1}{2\pi}\int_{\Delta(g_0,\dots,g_n)} \omega\right].$$

Lemma 5.7 (van Est)**.** *The map $I : \Omega^n(\mathcal{X})^G \to H^n_c(G;\mathbb{R})$ is an isomorphism.*

Proof. $\Omega^n(X)$ is an injective resolution of the trivial module; by Cartan's lemma the differentials in the complex $(\Omega^n(X)^G, d)$ are trivial. □

One can try to estimate the integrals appearing in the explicit van Est formula. In degree 2 (and only in degree 2), J. L. Dupont succeeded in doing so. Thereby he proved:

Proposition 5.8 (Dupont, [18])**.** *The cocycles in the image of the degree* 2 *van Est map are bounded.*

This yields in particular surjectivity of the comparison map in degree 2. For the state of the art in higher degree see [24]. Combining everything we said so far we obtain:

Corollary 5.9. *Let G be a simple Lie group with symmetric space $\mathcal{X} = G/K$. Then*

$$H^2_{cb}(G;\mathbb{R}) \cong H^2_c(G;\mathbb{R}) \cong \Omega^2(\mathcal{X})^G.$$

Definition 5.10. A simple Lie group is called *Hermitian* if $\Omega^2(\mathcal{X})^G \neq \{0\}$.

The geometry of Hermitian Lie groups is well-known. We only provide a few key-words:

- If G is a Hermitian simple Lie group, then $\dim \Omega^2(\mathcal{X})^G = 1$.
- Every $\omega \in \Omega^2(\mathcal{X})^G \setminus \{0\}$ is in fact a Käbler form on $\mathcal{X}$, i.e. there exists a complex structure $J : T\mathcal{X} \to T\mathcal{X}$ such that $g(v,w) := \omega(v, Jw)$ is a Riemannian metric. This complex structure is unique, so $\mathcal{X}$ is canonically a complex manifold. We refer to this complex manifold as a *Hermitian symmetric space.* We can normalize ω so that the minimal holomorphic sectional curvature of the Hermitian metric $H = g + i\omega$ is -1. With this normalization, ω becomes unique. We then denote by κ^G and κ^G_b the corresponding classes in $H^2_c(G;\mathbb{R})$ and $H^2_{cb}(G;\mathbb{R})$, called the *Kähler class*, respectively *bounded Käbler class* of G.

- Via the Harish-Chandra embedding theorem, every Hermitian symmetric space $\mathcal{X}$ is isomorphic as a complex manifold to a bounded domain $\mathcal{D} \subset \mathbb{C}^{\dim \mathcal{X}}$, which is unique up to unique isomorphism. The Riemannian metric g (and hence ω) can be recovered from $\mathcal{D}$ as the Bergman metric associated with the embedding into $\mathbb{C}^{\dim \mathcal{X}}$. A symmetric domain $\mathcal{D}$ isomorphic to a symmetric space is called a *bounded symmetric domain*. Thus Hermitian Lie groups are precisely the identity components of automorphism groups (i.e. groups of biholomorphisms) of bounded symmetric domains.
- A tube domain in $\mathbb{C}^N$ is a subdomain $T = \mathbb{R}^N + i\Omega$, where $\Omega \subset \mathbb{R}^N$ is an open convex sone. A bounded symmetric domain is *of tube type* it is biholomorphic to a tube domain. The prototype of a tube domain is the upper half plane; in particular, $\mathbb{D}$ is of tube type. We say that a simple Hermitian Lie group is of tube type if the associated bounded symmetric domain has this property. Thus $\mathrm{PU}(1,1)$ is of tube type.
- Other example of Hermitian Lie groups of tube type include the real-split symplectic groups $\mathrm{Sp}(2n)$ and the groups $SU(n,n)$. The groups $SU(p,q)$ for $p \neq q$ are examples of Hermitian Lie groups, which are *not* of tube type.

All the cohomological methods we have presented above use the second bounded cohomology. Thus, with these methods, we can only obtain results about representations into Hermitian Lie groups. For non-Hermitian Lie groups like $\mathrm{SL}_n(\mathbb{R})$ these methods are completely useless. In order to treat such groups, it would be necessary to obtain a better understanding of bounded cohomology in degrees > 2.

5.3. **The boundary resolution.** Consider again the example of $PU(1,1)$ acting on the Poincaré disc $(\mathbb{D}, \omega)$. We have constructed a family of (boundedly) cohomologous cocycles $c_{\omega,o}$ depending on some basepoint $o \in \mathbb{D}$. What happens if o wanders off to the boundary?

Recall that every triple $(\xi_0, \xi_1, \xi_2) \in (S^1)^3 = (\partial\mathbb{D})^3$ defines an ideal triangle $\Delta(\xi_0, \xi_1, \xi_2)$. Thus for $\xi \in S^1$ we can define a cocycle $c_{\omega,\xi}$ by the same formula

$$c_{\omega,\xi}(g_0, g_1, g_2) := \frac{1}{2\pi} \int_{\Delta(g_0.\xi, g_1.\xi, g_2.\xi)} \omega.$$

as before. However, something strange happens when passing to the boundary: Every ideal simplex has volume π! Thus, $c_{\omega,\xi}(g_0, g_1, g_2) = \frac{1}{2} \cdot \mathrm{o}(g_0.\xi, g_1.\xi, g_2.\xi)$, where

$$\mathrm{o}(\xi_0, \xi_1, \xi_2) = \begin{cases} +1, & (\xi_0, \xi_1, \xi_2) \text{ non-degenerate and positively oriented} \\ -1 & (\xi_0, \xi_1, \xi_2) \text{ non-degenerate and negatively oriented} \\ 0 & (\xi_0, \xi_1, \xi_2) \text{ degenerate.} \end{cases}$$

is the orientation cocycle of the circle. This cocycle is still Borel measurable and bounded, but no longer smooth. Now, by a theorem of D. Wigner, the cohomology of a Lie group computed from Borel measurable cochains is isomorphic to the cohomology computed from continuous cochains, in symbols

$$H^\bullet_B(G; \mathbb{R}) \cong H^\bullet_c(B; \mathbb{R}).$$

Under this isomorphism, the cocycles $c_{\omega,\xi}$ represent the Kähler class. Burger and Monod have proved that there is a similar isomorphism

$$H^\bullet_{Bb}(G; \mathbb{R}) \cong H^\bullet_{cb}(B; \mathbb{R}),$$

sending the class of $c_{\omega,\xi}$ to the bounded Kähler class. This is part of a more general theory, which we briefly sketch:

The space S^1 is the prototype of a *boundary* for the group $\mathrm{PU}(1,1)$. For the purposes of these notes let us call a standard Borel G-space B a *weak G-boundary* if it an amenable G-space in the sense of Zimmer and the action of G on $B \times B$ is ergodic. If you do not know what these words mean think of the following examples, which suffice for our purposes: If G is a simple Lie group and P is a minimal parabolic, then G/P is a weak G-boundary. Similarly, if $\Gamma < G$ is a lattice (e.g. a surface group in $\mathrm{PU}(1,1)$), then G/P is still a weak Γ-boundary. For Gromov-hyperbolic groups, the Gromov boundary is a weak boundary. With this terminology understood, the starting point of the Burger-Monod approach to (continuous) bounded cohomology can be formulated as follows:

Proposition 5.11 (Boundary resolution). *Let G be a locally-compact, 2nd countable group and B a weak G-boundary. Then*

$$0 \to \mathbb{R} \to L^\infty(B) \to L^\infty(B^2) \to L^\infty(B^3) \to \dots,$$

where all maps are given by the usual homogeneous differentials, is an injective augmented resolution of the trivial Banach G-module $\mathbb{R}$. Thus there is a canonical isomorphism

$$H^\bullet_{cb}(G;\mathbb{R}) \cong H^\bullet(0 \to L^\infty(B)^G \to L^\infty(B^2)^G \to L^\infty(B^3)^G \to \dots)$$

Exercise 5.12. Show that the boundary resolution is homotopic to the *alternating boundary resolution*

$$0 \to L^\infty_{alt}(B) \to L^\infty_{alt}(B^2) \to L^\infty_{alt}(B^3) \to \dots,$$

where $L^\infty_{alt}(B^n) \subset L^\infty(B^n)$ denotes the subspace of alternating functions.

Let us specialize to degree 2: We have

$$H^2_{cb}(G;\mathbb{R}) = \frac{\mathcal{Z}^2_{alt}(B)^G}{\mathcal{B}^2_{alt}(B)^G} := \frac{\ker(d : L^\infty_{alt}(B^3)^G \to L^\infty_{alt}(B^4)^G)}{\mathrm{Im}(d : L^\infty_{alt}(B^2)^G \to L^\infty_{alt}(B^3)^G)}.$$

Now, since G acts ergodically on B^2, every G-invariant function on B^2 is constant, hence every alternating G-invariant function on B^2 is 0. Thus there are no coboundaries in degree 2 and we have:

Proposition 5.13. $H^2_{cb}(G;\mathbb{R}) = \mathcal{Z}^2_{alt}(B)^G = \{c \in L^\infty_{alt}(B^3)^G \mid dc = 0\}$.

This is the reason why the boundary resolution is very convenient for computations in degree 2.

5.4. **Functoriality and transfer.** There is one major problem with the boundary resolution, which concerns functoriality. Assume we are given two Lie groups G_1 and G_2 with respective weak boundaries B_1 and B_2 and a cohomology class $\alpha \in H^2_{cb}(G_2;\mathbb{R})$. How to pull back α via a homomorphism $\rho : G_1 \to G_2$. It is not completely trivial that there exists a unique equivariant map $\varphi : B_1 \to B_2$, although this is in fact the case by a result of Furstenberg (uniqueness is up to measure zero). However, even in the most favorable situations, the image of B_1 in B_2 will typically be a null set (unless B_1 and B_2 have the same dimension). However, it is not possible to restrict an L^∞-function (which is actually a function class, of course) to a null set. Therefore we cannot implement the pullback via φ. There is no solution to this problem, only work-arounds. The most useful one is due to Burger and Iozzi [4]:

Lemma 5.14 (Functoriality redeemed)**.** *Let G_1, G_2 be locally compact groups with respective weak boundaries B_1, B_2. Let $c : B_2^3 \to \mathbb{R}$ be a bounded, alternating, G-invariant measurable function with $dc(\xi_0, \ldots, \xi_3) = 0$ for all (not just almost all!) triples $(\xi_0, \ldots, \xi_3) \in B^3$. Then c defines a function class in $\mathcal{Z}^2_{alt}(B)^G$, which in turn defines a class $\alpha \in H^2_{cb}(G_2; \mathbb{R})$. Let $\rho : G_1 \to G_2$ be a continuous homomorphism and $\varphi : B_1 \to B_2$ be an equivariant measurable map. Then the class defined by $\varphi^* c$ in $H^2_{cb}(G_1; \mathbb{R})$ coincides with $\rho^*\alpha$.*

Note that in general there is no guarantee that a given cocycle can be represented by a function c as above. Fortunately, in all known examples of bounded cohomology classes on a boundary there is such a representative. (There are however examples, due to Bucher and Monod, of L^∞ cocycles on a flag variety associated with a non-minimal parabolic, which cannot be represented by a measurable function as above on that specific flag variety. This does not exclude the possibility that every boundary cocycle has a representative as above, but it shows that if there is a general principle, then it is necessarily quite subtle.)

One situation, where the above problems do not occur is when $\rho : \Gamma \to G$ is the inclusion of a lattice. In this case, ρ^* is realized by the inclusion map

$$\mathcal{Z}^2_{alt}(B)^G \to \mathcal{Z}^2_{alt}(B)^\Gamma.$$

In particular, the restriction map $H^2_{cb}(G; \mathbb{R}) \to H^2_b(\Gamma; \mathbb{R})$ is injective. The map $T^2_b : Z^2_{alt}(B)^\Gamma \to \mathcal{Z}^2_{alt}(B)^G$ given by

$$T^2_b f(x, y, z) := \int_{\Gamma\backslash G} f(gx, gy, gz) dg$$

yields an explicit left-inverse of this restriction map. By abuse of notation we also write T^2_b for the induced map $T^2_b : H^2_b(\Gamma; \mathbb{R}) \to H^2_{cb}(G; \mathbb{R})$, which is called the *bounded transfer map* in degree 2.

5.5. **The generalized Maslov index and the tube type dichotomy.** Let G be a Hermitian simple Lie group of tube type and let $\kappa^G_{(b)}$ denote its (bounded) Kähler class. We know from the last section that there exists a unique bounded alternating cocycle $c : (G/P)^3 \to \mathbb{R}$ representing κ^G_b, where P is a minimal parabolic of G. The G-equivariant quotients of G/P are precisely the flag varieties G/Q, where Q is a parabolic containing P. Given such a parabolic Q there may or may not exist a cocycle $\bar{c} : (G/Q)^3 \to \mathbb{R}$ representing κ^G_b. If it does exists, then it is clearly unique (since already c is). In this case we say that κ^G_b can be represented on G/Q. It turns out that bounded Kähler classes can be represented on a flag variety associated with a distinguished maximal parabolic subgroup. We are now going to describe this distinguished flag variety geometrically.

As mentioned earlier, we can always realize G as the groups of biholomorphisms of a bounded symmetric domain $\mathcal{D} \subset \mathbb{C}^N$. An example to keep in mind is given by the matrix ball

$$\mathcal{D}_n := \{X \in M_n(\mathbb{C}) \mid M \text{ symmetric}, \mathbf{1} - M^*M \text{ positive definite}\},$$

for which we have $\mathrm{Aut}(\mathcal{D}_n) \cong \mathrm{PSp}(2n)$, the action being by generalized fractional linear transformations.

Exercise 5.15. Show that the matrix ball $\mathcal{D}_n$ is of tube type by providing an explicit biholomorphism with the Siegel upper half plane

$$\mathcal{H}_n := \{X + iY \in \mathrm{Sym}_n(\mathbb{C}) \mid X, Y \in \mathrm{Sym}_n(\mathbb{R}), Y \text{ positive definite}\}.$$

[Hint: For $n = 1$ the biholomorphism is the Cayley transform. Now observe that the explicit formula for the Cayley transform makes sense also for matrices.]

We now return to the general case: The topological boundary $\partial\mathcal{D} := \overline{\mathcal{D}} \setminus \mathcal{D}$ depends on the concrete realization $\mathcal{D}$. However, there is always a unique closed G-orbit in $\mathcal{D}$, which is independent of the realization and called the *Shilov boundary* $\check{S}$ of G. It is the smallest compact subset of $\partial\mathcal{D}$ such that the maximum principle holds, i.e.

$$\sup_{z \in \mathcal{D}} f(z) = \sup_{z \in \check{S}} f(z).$$

for every continuous function f on $\overline{\mathcal{D}}$, which is holomorphic in $\mathcal{D}$.

Example 5.16. In the case of the matrix ball, the topological boundary is formed by all complex symmetric matrices M for which $\mathbf{1} - M^*M$ is positive semi-definite, but not positive definite. The G-orbits inside this topological boundary are given by those matrices of a constant rank r with $0 \leq r \leq n-1$, i.e. there are n different G-orbits. The unique closed G-orbit, i.e. the Shilov boundary, is simply given by the rank 0 component

$$\check{S}_n := \mathrm{Sym}_n(\mathbb{C}) \cap U(n).$$

Returning to the general case we observe that since $\check{S}$ is closed in $\overline{\mathcal{D}}$ it is always a compact homogeneous G-space, i.e. a flag variety. Thus there exists a maximal parabolic Q (unique up to conjugation) such that

$$\check{S} \cong G/Q.$$

We refer to Q and its conjugates as *Shilov parabolics*. One can now try to proceed as in the $\mathrm{PU}(1,1)$-case and to push the Kähler cocycle $c_{\omega,o}$ to the Shilov boundary by letting $o \to \xi \in \check{S}$. It is not at all obvious that such an approach can work, and indeed one has to be very careful from which directions one approaches the Shilov boundary, in particularly for non-transversal triples of points. In any case, it is possible to construct a limiting cocycle $\beta_{\check{S}}$. For $\mathrm{PU}(1,1)$ we have seen that $\beta_{\check{S}}$ is a multiple of the orientation cocycle. In the case of the symplectic groups $\mathrm{Sp}(2n)$, the cocycle $\beta_{\check{S}}$ is also a classical object of symplectic geometry, namely the so-called *Maslov index*. For general Hermitian G, the *generalized Maslov index* $\beta_{\check{S}}$ was constructed by Clerc [13]. Once the generalized Maslov index is constructed, it is not too hard to see that it does realize the bounded Kähler class:

Proposition 5.17 (Shilov realizability, [13])**.** *Let G be a simple Lie group of Hermitian type, $\check{S}$ the associated Shilov boundary and κ_b^G the bounded Kähler class.*

(i) *The generalized Maslov index $\beta_{\check{S}}$ is a bounded measurable, alternating G-invariant function.*
(ii) *The generalized Maslov index is continuous on the set $\check{S}^{(3)}$ of pairwise transverse triples and satisfies $d\beta_{\check{S}} = 0$ pointwise.*
(ii) *The bounded Kähler class κ_b^G can be represented on $\check{S}$, and the unique representing cocycle is precisely the generalized Maslov index.*

We emphasize that in order to know the bounded cohomology class defined by the Maslov index it is enough to know $\beta_{\check{S}}$ on the generic set $\check{S}^{(3)}$. However, in order to use Lemma 5.14 one needs to know that $\beta_{\check{S}}$ extends to a pointwise measurable cocycle on all of $\check{S}^3$. This is the most difficult part of Proposition 5.17. On the other hand, the restriction of $\beta_{\check{S}}$ to the generic set $\check{S}^{(3)}$ is very-well understood. It was first described by Clerc and Ørsted in [14]. In [6] Burger, Iozzi and Wienhard provide a formula in terms of what they call the Hermitian triple product. A consequence of this formula is the following dichotomy:

Proposition 5.18 (Bounded Kähler class dichotomy, [6])**.** (i) *If G is of tube type, then $\beta_{\check{S}}|_{\check{S}^{(3)}}$ is locally constant and thus takes only a finite set of rational values.*
(ii) *If G is not of tube type, then $\beta_{\check{S}}|_{\check{S}^{(3)}}$ takes a continuum of values.*

5.6. **Relation to bounded Euler class.** As pointed out by Iozzi in [25] one can use the boundary resolution to give a simple proof of the following fact, which will be needed below:

Proposition 5.19. *Let $\iota : \mathrm{PU}(1,1) \to \mathrm{Homeo}^+(S^1)$ denote the standard embedding. Then $\iota^* e_b^{\mathbb{R}} = \kappa_b^{\mathrm{PU}(1,1)}$.*

Proof. Let $H := \mathrm{Homeo}^+(S^1)$. Let $c : H^3 \to \mathbb{R}$ be the cocycle given by $c(h_0, h_1, h_2) = o(h_0.1, h_1.1, h_2.1)$. In view of the boundary model it then suffices to show that c is boundedly cohomologous to the standard representative e_σ of $e_b^{\mathbb{R}}$. In fact we claim that

$$e_\sigma - c = d\beta, \tag{5.1}$$

where

$$\beta(f, g) = \begin{cases} -\frac{1}{2} & f(1) \neq 1, \\ 0 & \text{otherwise.} \end{cases}$$

□

Exercise 5.20. Show the equality (5.1). (If you get stuck, you may want to consult [25]).

Note that as a consequence we also get $\iota^* e^{\mathbb{R}} = \kappa^{\mathrm{PU}(1,1)}$ for free, using naturality of the comparison map.

6. Applications to Shilov-Anosov representations

6.1. **Shilov-Anosov representations and bounded Kähler class.** We pause for a moment to give a first application of the material of the last section. Let G be a Hermitian Lie group of tube type, Q a Shilov-parabolic and Q^- a parabolic opposite G. It turns out that Q and Q^- are conjugate, hence $G/Q \cong G/Q^- \cong \check{S}$. Assume now that $\rho : \Gamma_g \to G$ is a (Q, Q^-)-Anosov-representation, or Shilov-Anosov representation for short. Using our machinery we prove:

Proposition 6.1. *Let $\rho : \Gamma_g \to G$ be a Shilov-Anosov representation. Assume that G is of tube type. Then there exists $\lambda \in \mathbb{Q}$ such that $\rho^* \kappa_b^G = \lambda \cdot \kappa_b^{\Sigma_g}$.*

Proof. By the Guichard-Wienhard characterization of Anosov representations (see [17]) there exists a continuous ρ-equivariant map $\varphi : S^1 \to \check{S}$ which maps $(S^1)^{(3)}$ to $\check{S}^{(3)}$. By Lemma 5.14 and (the hard part of) Proposition 5.17 the class $\rho^*\kappa_b^G$ is represented by the cocycle $\varphi^*\beta_{\check{S}}$. Now

$$\varphi^*\beta_{\check{S}}|_{(S^1)^{(3)}} \to \mathbb{Q}$$

is continuous with discrete image (by the bounded Kähler class dichotomy), thus locally constant. However, the only alternating cocycles on S^1, which are locally constant on $(S^1)^{(3)}$ are the multiples of the orientation cocycle. We deduce that $\varphi^*\beta_{\check{S}} = \lambda \cdot o$, and since the left-hand side is rational almost everywhere, so is the right hand side. □

6.2. **The Toledo invariant.** We would like to combine Proposition 6.1 and Theorem 4.11 to obtain discreteness and faithfulness of Shilov-Anosov representations. However, for this we need to know that the constant λ appearing in Proposition 6.1 is non-zero. We thus need to understand the meaning of this constant.

There is a special relevance to classes in $H^2(\Gamma_g; \mathbb{R})$. due to the fact that surfaces are two-dimensional. Namely, $H_2(\Sigma_g; \mathbb{R}) = \mathbb{R} \cdot [\Sigma_g]$, where $[\Sigma_g]$ is the fundamental class. Thus we have a canonical evaluation map

$$H^2(\Gamma_g; \mathbb{R}) \cong H^2(\Sigma_g; \mathbb{R}) \to \mathbb{R}, \quad \alpha \mapsto \langle \alpha, [\Sigma_g] \rangle.$$

Consider now the special case of κ^{Σ_g}. By Proposition 5.19 we have $\kappa^{\Sigma_g} = \kappa^{\mathrm{PU}(1,1)}|_{\Gamma_g}$; thus in the de Rham model, κ^{Σ_g} is represented by $\frac{1}{2\pi}$ times the hyperbolic volume form. We deduce that

$$\langle \kappa^{\Sigma_g}, [\Sigma_g] \rangle = \frac{1}{2\pi} \int_{\Sigma_g} d\mathrm{vol} = |\chi(\Sigma_g)| = 2 - 2g.$$

We conclude that if $\rho : \Gamma_g \to G$ is a representation with

$$\rho^*\kappa_b^G = \lambda \cdot \kappa_b^{\Sigma_g}, \tag{6.1}$$

then

$$\rho^*\kappa^G = \lambda \cdot \kappa^{\Sigma_g},$$

and hence

$$\langle \rho^*\kappa^G, [\Sigma_g] \rangle = \lambda \cdot \langle \kappa^{\Sigma_g}, [\Sigma_g] \rangle = \lambda \cdot |\chi(\Sigma_g)|,$$

i.e.

$$\lambda = \frac{\langle \rho^*\kappa^G, [\Sigma_g] \rangle}{|\chi(\Sigma_g)|}. \tag{6.2}$$

Definition 6.2. Let G be a Hermitian simple group, κ^G its Kähler class, Γ_g a surface group and $\rho : \Gamma_g \to G$ a representation. Then the number

$$T(\rho) := \langle \rho^*\kappa^G, [\Sigma_g] \rangle \in \mathbb{Q}$$

is called the *Toledo invariant* of the representation ρ.

Note that $T(\rho)$ takes a discrete set of rational values, since κ^G is rational. Moreover, as a consequence of (6.2) we see that λ as given in (6.1) vanishes if and only if the Toledo invariant vanishes. Combining this with Proposition 6.1 and Theorem 4.11 we have obtained a complete proof of the following result:

Theorem 6.3. *Let $\rho : \Gamma_g \to G$ be a Shilov-Anosov representation into a Hermitian Lie group of tube type with non-zero Toledo invariant. Then ρ is faithful and its image is discrete.*

Of course, as we have already seen in [17, 31], much more is true: Any Anosov representation is essentially discrete and faithful. However, the proof given here is still interesting in its own right, since one can push its idea quite a bit further: Firstly, the assumption that G be of tube type is actually unnecessary, since one can show that every Shilov-Anosov representation with non-zero Toledo invariant preserves a subdomain of tube type. Secondly, and more importantly, we manage in [1] to extend the result to cover also the following case:

Theorem 6.4. *Let G be a Hermitian Lie group and $\rho : \Gamma_g \to G$ a representation of non-zero Toledo invariant which is a limit of Shilov-Anosov representations. Then ρ is faithful and its image is discrete.*

Using the theorem we provide new examples of discrete, faithful representations which are not Anosov. The following problem is open to the best of my knowledge: Consider a discrete and faithful representation $\rho : \Gamma_g \to \mathrm{Sp}(4)$, which is Zariski-dense and of non-zero Toledo invariant. Is it true that (6.1) holds for some $\lambda \in \mathbb{R}$? Representations satisfying this condition are called *weakly maximal*. So far we do not know any other discrete faithful representations, but this is probably mainly because we have no other tools to detect discreteness and faithfulness than the cohomological ones explained here.

7. Bounded cohomology III: The Gromov seminorm

7.1. **Motivation: Boundedness of the Toledo-invariant.** We have seen in the last section that the Toledo-invariant of a representation of a surface group into a Hermitian simple Lie group yields valuable information. We observe:

Proposition 7.1. *For a given Hermitian simple Lie group G, the Toledo invariant $T : \mathrm{Rep}(\Gamma_g; G) \to \mathbb{R}$ takes only finitely many values.*

Proof. It is not hard to see that T is continuous; since it range is discrete, it is thus locally constant. However, $\mathrm{Rep}(\Gamma_g; G)$ as an algebraic variety has only finitely many connected components. □

We thus know that the supremum $C_{G,\Gamma_g} := \sup_\rho |T(\rho)|$ taken over the representation variety $\mathrm{Rep}(\Gamma_g; G)$ is finite - can we determine its value? Since the range of the Toledo invariant is finite, the supremum is actually attained - but for which representations? We will answer these type of questions using bounded cohomology.

7.2. **The Gromov seminorm.** Gromov's work on bounded cohomology was to a large extend based on the fundamental observation that the cohomology of a cocomplex of Banach spaces with continuous codifferentials carries a canonical semi norm, and on exploiting such seminorms geometrically. More concretely, consider the standard homogeneous resolution

$$0 \to C_b^0(G)^G \to C_b^1(G)^G \to \dots$$

The space $C_b^n(G)$ are Banach spaces when equipped with the sup-norm, hence so are there closed(!) subspaces $C_b^n(G)^G$ and $\mathcal{Z}C_b^n(G)^G$. Unfortunately the subspace of coboundaries need not be closed in this space, whence $H_b^n(G;\mathbb{R})$ is in general not Hausdorff with respect to the quotient topology, let alone a Banach space. The best we can do is to consider the seminorm

$$\|\alpha\| := \inf\{\|c\|_\infty \mid c \in \alpha\}$$

on $H_b^n(G;\mathbb{R})$. Another drawback is that this semi norm is not canonical: We can easily write down injective resolution, which lead to a different semi norm on the level of cohomology. Indeed, it suffices to rescale the differentials. Despite all these shortcomings, the seminorm $\|\cdot\|$ is extremely useful. It is often referred to as the *Gromov seminorm*, although Gromov was by no means the first to consider it. (However, he was probably the first to provide substantial applications.)

One immediate property of the Gromov semi norm, which is extremely useful, is the following:

Proposition 7.2. *For every $\alpha \in H_{cb}^\bullet(G;\mathbb{R})$ and every $\rho : H \to G$ we have*

$$\|\rho^*\alpha\| \leq \|\alpha\|.$$

As we said, if we replace the homogeneous bar resolution by another injective resolution, then the induced isomorphism is in general not isometric. However, there are a couple of nice injective resolutions, which do indeed compute the same seminorm. We refer to these as *isometric resolutions*. For us the key example is as follows:

Proposition 7.3 (Burger-Monod, [9])**.** *The boundary resolution and the alternating boundary resolution are isometric.*

As an immediate consequence we deduce:

Corollary 7.4. *The Gromov seminorm on $H_b^2(G;\mathbb{R})$ is a norm for any locally-compact, second countable group G.*

The following section exploits some less obvious consequences of Proposition 7.3.

7.3. **The Gromov seminorm and transfer.** Throughout this section let $G = PU(1,1)$ and $\Gamma \cong h(\Gamma_g)$ be the image of Γ_g under a hyperbolization h, i.e. a cocompact lattice in G. We have seen above that the restriction map $H_{cb}^2(G;\mathbb{R}) \to H_b^2(\Gamma;\mathbb{R})$ has a left-inverse $T_b^2 : H_b^2(\Gamma;\mathbb{R}) \to H_{cb}^2(G;\mathbb{R})$ given by bounded transfer. From the explicit formula for T_b^2 we deduce:

Lemma 7.5. *The bounded transfer map $T_b^2 : H_b^2(\Gamma;\mathbb{R}) \to H_{cb}^2(G;\mathbb{R})$ is norm non-increasing, i.e. $\|T_b\alpha\| \leq \|\alpha\|$.*

We also recall that $H_{cb}^2(G;\mathbb{R}) = \mathbb{R}\cdot\kappa_b^G$, where κ_b^G is represented by $\frac{1}{2}\cdot o$ in the boundary resolution. We draw two consequences: Firstly, for every $\alpha \in H_b^2(\Gamma;\mathbb{R})$ there exists $\tau(\alpha) \in \mathbb{R}$ such that

$$T_b\alpha = \tau(\alpha)\cdot\kappa_b^G. \tag{7.1}$$

Secondly, we have $\|\kappa_b^G\| = \|\frac{1}{2}\cdot o\|_\infty = \frac{1}{2}$. We observe:

Proposition 7.6. *For every $\alpha \in H_b^2(G;\mathbb{R})$ we have*

$$|\tau(\alpha)| \leq 2\|\alpha\|.$$

Equality holds if and only if α is a multiple of $\kappa_b^{\Sigma_g}$.

Proof (cf. [5]*).* In view of Lemma 7.5 we have

$$\|\alpha\| \geq \|T_b\alpha\| = \left\| \frac{\tau(\alpha)}{2} \cdot o \right\| = \frac{1}{2} \cdot |\tau(\alpha)|,$$

which gives the desired inequality. Now assume that equality holds. We may assume without loss of generality that $\tau(\alpha) \geq 0$. Let c be a cocycle representing α in the boundary resolution. Then $\tau(\alpha) = 2\|c\|_\infty$, and thus

$$\|c\|_\infty \cdot o(x,y,z) = \tau(\alpha) \cdot \frac{o(x,y,z)}{2} = T_b^2 c(x,y,z) = \int_{\Gamma\backslash G} c(gx,gy,gz)dg,$$

which by G-invariance of β we can rewrite as

$$\int_{\Gamma\backslash G} \|c\|_\infty \cdot o(gx,gy,gz) - c(gx,gy,gz)dg = 0.$$

Assume now that the triple (x,y,z) is positively oriented. Then the above formula specializes to

$$\int_{\Gamma\backslash G} \|c\|_\infty - c(gx,gy,gz)dg = 0,$$

which implies $c(gx,gy,gz) = \|c\|_\infty$ for almost all g. Since g acts 3-transitively, this means that c is constant on positively oriented generic triples; combined with the antisymmetry it follows that c is almost everywhere a multiple of the orientation cocycle as claimed. □

We can actually express the number $\tau(\alpha)$ more explicitly:

Proposition 7.7. *For every $\alpha \in H_b^2(\Gamma;\mathbb{R})$ we have*

$$\tau(\alpha) = \frac{\langle c_\Gamma^2(\alpha), [\Sigma]\rangle}{|\chi(\Sigma_g)|}$$

Proof. There is a natural transfer map $T^2 : H^2(\Gamma;\mathbb{R}) \to H_c^2(G;\mathbb{R})$ in usual (continuous) cohomology, and the comparison map intertwines this with the bounded transfer discussed above. In particular, $T(c_\Gamma^2(\alpha)) = \tau(\alpha) \cdot \kappa^G = T(\tau(\alpha) \cdot \kappa^{\Sigma_g})$. Now since $\dim H^2(\Gamma;\mathbb{R}) = \dim H_c^2(G;\mathbb{R}) = 1$, the transfer map is an isomorphism, whence $c_\Gamma^2(\alpha) = \tau(\alpha) \cdot \kappa^{\Sigma_g}$, and thus

$$\langle c_\Gamma^2(\alpha), [\Sigma]\rangle = \tau(\alpha) \cdot \langle \kappa^{\Sigma_g}, [\Sigma]\rangle = \tau(\alpha) \cdot |\chi(\Sigma_g)|.$$

□

Combining both propositions we deduce:

Corollary 7.8. *Let $\alpha \in H_b^2(\Gamma;\mathbb{R})$. Then*

$$\langle c_\Gamma^2(\alpha), [\Sigma]\rangle \leq 2 \cdot |\chi(\Sigma_g)| \cdot \|\alpha\|$$

with equality if and only if α is a multiple of κ_b^G.

Note that the proof of the inequality in Corollary 7.8 is essentially trivial: All the work went into the understanding of the equality case.

8. Maximal representations

8.1. **Generalized Milnor-Wood inequality and maximal representations.** Throughout this section let G be a Hermitian simple Lie group and let $\rho : \Gamma_g \to G$ be a representation. The following result is due to Milnor [28][5] in the case $G = \mathrm{PU}(1,1)$ and due to Burger, Iozzi and Wienhard [8] in the general case.

Theorem 8.1 (Generalized Milnor-Wood inequality)**.** *The Toledo invariant satisfies*

$$T(\rho) \leq |\chi(\Sigma_g)| \cdot \mathrm{rk}(G).$$

Equality holds if and only if the following two conditions are satisfied:

(i) $\rho^*\kappa_b^G = \mathrm{rk}(G) \cdot \kappa_b^{\Sigma_g}$.
(ii) $\|\rho^*\kappa_b^G\| = \|\kappa_b^G\|$.

Proof. Combining Corollary 7.8 and Proposition 7.2 We have the chain of inequalities

$$\begin{aligned} T(\rho) &= \langle \rho^*\kappa_b^G, [\Sigma]\rangle \\ &\leq 2 \cdot |\chi(\Sigma_g)| \cdot \|\rho^*\kappa_b^G\| \\ &\leq 2 \cdot |\chi(\Sigma_g)| \cdot \|\kappa_b^G\| \\ &= 2 \cdot |\chi(\Sigma_g)| \cdot \|\kappa_b^G\|, \end{aligned}$$

Now the generalized Milnor-Wood inequality follows from the formula

$$\|\kappa_b^G\| = \frac{1}{2} \cdot \mathrm{rk}(G),$$

which is due to Clerc and Ørsted [15] in general and Domic and Toledo [16] for the classical groups. Equality in the generalized Milnor-Wood inequality holds if and only the two equalities $\langle \rho^*\kappa_b^G, [\Sigma]\rangle = 2 \cdot |\chi(\Sigma_g)| \cdot \|\rho^*\kappa_b^G\|$ and $\|\rho^*\kappa_b^G\| = \|\kappa_b^G\|$ hold. By Corollary 7.8 the former equality is equivalent to the existence of $\lambda \in \mathbb{R}$ wit $\rho^*\kappa_b^G = \lambda \cdot \kappa_b^{\Sigma_g}$, and then (ii) implies $\lambda = \mathrm{rk}(G)$. □

Motivated by this inequality we define:

Definition 8.2. A representation $\rho : \Gamma_g \to G$ is called a *representation of maximal Toledo invariant* (or *maximal representation* for short) if

$$T(\rho) = |\chi(\Sigma_g)| \cdot \mathrm{rk}(G).$$

It is called *weakly maximal* if there exists $\lambda \geq 0$ such that $\rho^*\kappa_b^G = \lambda \cdot \kappa_b^{\Sigma_g}$ and *tight* if $\|\rho^*\kappa_b^G\| = \|\kappa_b^G\|$.

Corollary 8.3. *A representation is maximal if and only if it is weakly maximal and tight.*

Combining this with Theorem 4.11 we deduce:

Theorem 8.4. *Every maximal representation (in fact, every weakly maximal representation of non-zero Toledo invariant) is discrete and faithful.*

In fact, maximal representations are Shilov-Anosov. We will not prove this here, though.

[5]There is a corresponding result in the non-orientable case due to Wood [30].

8.2. **Goldman's characterization of hyperbolizations.** Specializing to the case $G = \mathrm{PU}(1,1)$ we obtain:

Theorem 8.5. *Let $\rho : \Gamma_g \to \mathrm{PU}(1,1)$ be a representation. Then the following are equivalent:*

(i) *ρ is maximal.*
(ii) *$T(\rho) = |\chi(\Sigma)|$.*
(iii) *$\rho^*\kappa_b^G = \kappa_b^{\Sigma_g}$.*
(iv) *$\exists \lambda > 0$: $\rho^*\kappa_b^G = \lambda \cdot \kappa_b^{\Sigma_g}$.*
(v) *ρ is discrete, faithful and orientation-preserving.*
(vi) *ρ is a hyperbolization.*

Proof. The implications (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(v) have been discussed above in greater generality. The implication (v) $\Rightarrow$ (vi) is a classical result of Dehn, Nielsen and Baer, and (vi) $\Rightarrow$ (iii) holds by definition. □

The equivalence (i)$\Leftrightarrow$(vi) was first established by Goldman in his thesis [21]. This was one of the starting points of higher Teichmller theory in general, and the theory of maximal representations in particular.

8.3. **Maximal representations and Higgs bundles.** One reason for the broad interest in the theory of maximal representations is the fact that they give rise to Higgs bundles, hence are amenable to harmonic map techniques. Given a representation $\rho : \Gamma_g \to G$ there is always a unique homotopy class of equivariant maps $\mathbb{D} \to \mathcal{X} = G/K$. However, this homotopy class of maps admits a harmonic representative if and only if the Zariski closure of $\rho(\Gamma_g)$ in G is a reductive subgroup. Therefore the following result of Burger, Iozzi and Wienhard is crucial for the theory of Higgs bundles:

Theorem 8.6 ([7])**.** *Maximal, and in fact all tight representations have reductive Zariski closure.*

The proof is another example for the use of cohomological techniques. In view of the cohomological definition of tight representations this is not surprising. In any case, this is a story for another time.

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