CONVEX COCOMPACT GROUPS IN REAL RANK ONE AND HIGHER

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Let G be a real, connected, noncompact, semisimple, linear Lie group. The goal of this talk is to recall the classical notion of a convex cocompact subgroup of G when G has real rank 1 and to discuss possible generalizations to higher real rank, following Kleiner–Leeb [KL2] and Quint [Q].

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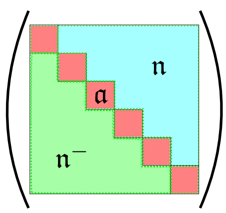
1. Preliminary notation

In the whole talk, K denotes a maximal compact subgroup of G, so that $X = G/K$ is the Riemannian symmetric space of G, and A denotes a maximal split connected semisimple abelian subgroup of G such that the Cartan decomposition $G = KAK$ holds; by definition, the real rank of G is $\mathrm{rank}_{\mathbb{R}}(G) = \dim(A)$. Consider the decomposition of the Lie algebra $\mathfrak{g}$ of G into joint eigenspaces for the adjoint action of A:

$$\mathfrak{g} = (\mathfrak{m} + \mathfrak{a}) + \mathfrak{n} + \mathfrak{n}^-, \tag{1.1}$$

where $\mathfrak{a}$ is the Lie algebra of A, where $\mathfrak{m}$ is the Lie algebra of the centralizer of A in K, and where $\mathfrak{n}$ (resp. $\mathfrak{n}^-$) is the sum of the positive (resp. negative) root spaces with respect to some fixed choice of positive restricted root system. Let M be the centralizer of A in K, with Lie algebra $\mathfrak{m}$, and let N (resp. N^-) be the image of $\mathfrak{n}$ (resp. $\mathfrak{n}^-$) under the exponential map $\exp : \mathfrak{g} \to G$. The group MA is the centralizer $Z_G(A)$ of A in G. The groups $P := (MA) \ltimes N$ and $P^- := (MA) \ltimes N^-$ are opposite minimal parabolic subgroups of G, with respective unipotent radicals N and N^-.

Example 1.1. For $G = \mathrm{SL}_m(\mathbb{R})$, we may take K to be $\mathrm{SO}(m)$ and A to be the group of diagonal matrices in G with positive entries. The Lie algebra $\mathfrak{g}$ is the set of traceless $m \times m$ real matrices, $\mathfrak{a}$ is the subset of diagonal matrices, and $\mathfrak{m}$ is trivial (G is split). With the usual choice of positive root system, $\mathfrak{n}$ (resp. $\mathfrak{n}^-$) is the set of upper (resp. lower) triangular nilpotent matrices.



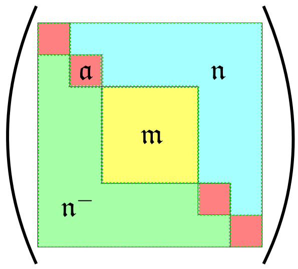
Example 1.2. For $G = \mathrm{SO}(p,q)_0$, the group K is isomorphic to $\mathrm{SO}(p)\times\mathrm{SO}(q)$. Suppose $p \leq q$. If we see G as the subgroup of $\mathrm{SL}_{p+q}(\mathbb{R})$ preserving the quadratic form

$$2\,x_1\,x_{p+q} + 2\,x_2\,x_{p+q-1} + \cdots + 2\,x_p\,x_{q+1} - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_q^2,$$

then we may take $\mathfrak{a}$ to be the set of diagonal matrices

$$\mathrm{diag}(t_1, \ldots, t_p, 0, \ldots, 0, -t_p, \ldots, -t_1)$$

with $t_1, \ldots, t_p \in \mathbb{R}$. In this case $\mathfrak{m}$ is the set of block diagonal matrices $\mathrm{diag}(0, D, 0)$ with $D \in \mathfrak{so}(q-p)$, with the two 0 blocks of size $p \times p$. With the usual choice of positive restricted root system, $\mathfrak{n}$ (resp. $\mathfrak{n}^-$) consists of strictly upper (resp. lower) triangular block matrices.



2. Convex cocompactness in real rank 1

2.1. **Definitions, examples, and characterizations.** In this section we assume that G has real rank 1. Recall the following classical definition.

Definition 2.1 ($\mathrm{rank}_{\mathbb{R}}(G) = 1$)**.** A discrete subgroup Γ of G (or any injective representation $\rho : \Gamma_0 \to G$ of a discrete group Γ_0 into G with image Γ) is said to be *convex cocompact* if one of the following equivalent conditions holds:

(1) Γ preserves and acts cocompactly on some nonempty convex subset $\mathcal{C}$ of $X = G/K$;
(2) Γ acts cocompactly on the closure F_Γ of the union of the translation axes of the hyperbolic elements of Γ in X (in other words, the closure of the union of all closed geodesics is compact in $\Gamma\backslash X$);
(3) the conservative set of the geodesic flow on the unit tangent bundle $T^1(\Gamma\backslash X)$ is compact.

By definition, the conservative set of the geodesic flow $(\varphi_t)_{t\in\mathbb{R}}$ is the closure of the set of vectors $v \in T^1(\Gamma\backslash X)$ such that $\varphi_t(v)$ returns to a compact subset of $T^1(\Gamma\backslash X)$ infinitely often.

Since $\mathrm{rank}_{\mathbb{R}}(G) = 1$, the Riemannian symmetric space X is Gromov-hyperbolic; its visual boundary identifies with G/P. Any nonempty, Γ-invariant, closed, convex subset of X must contain the convex hull $\mathrm{Conv}(\Lambda_\Gamma) \subset X$ of the limit set $\Lambda_\Gamma \subset G/P$ of Γ, hence (1) is equivalent to the fact that Γ acts cocompactly on $\mathrm{Conv}(\Lambda_\Gamma)$. The set F_Γ is the union of all geodesics of X with both endpoints in Λ_Γ; it is contained in $\mathrm{Conv}(\Lambda_\Gamma)$, so (1) immediately implies (2). The following classical lemma (see [Gd, Lem. 4] for instance) explains why (2) implies (1).

Lemma 2.2. *The set* $\mathrm{Conv}(\Lambda_\Gamma)$ *is contained in a uniform neighborhood of its subset* F_Γ.

Convex cocompact groups in real rank 1 include uniform lattices and Schottky groups (with no parabolic element). Other interesting examples are given by quasi-Fuchsian representations of surface groups into $\mathrm{PSL}_2(\mathbb{C})$.

We shall recall a proof of the following well-known lemma.

Lemma 2.3. *If* $\mathrm{rank}_{\mathbb{R}}(G) = 1$*, then the following conditions are equivalent:*

(*i*) Γ *is convex cocompact,*
(*ii*) Γ *is finitely generated and quasi-isometrically embedded in* G*,*
(*iii*) Γ *is Gromov-hyperbolic and there exists a continuous, injective, and* Γ*-equivariant map* $\xi : \partial_\infty\Gamma \to G/P = \partial_\infty X$*,*
(*iv*) Γ *is Gromov-hyperbolic and the natural inclusion of* Γ *in* G *is a* P*-Anosov representation.*

Recall that "Γ quasi-isometrically embedded in G" means that for some (hence any) finite generating subset S of Γ and some (hence any) G-invariant Riemannian metric d_G on G, there exist $c, C > 0$ such that for any $\gamma \in \Gamma$,

$$d_G(e, \gamma) \geq c\,\ell_S(\gamma) - C,$$

where $\ell_S : \Gamma \to \mathbb{N}$ is the word length function with respect to S. We shall recall the definition of Anosov representations in Section 2.3 below. The implication $(iv) \Rightarrow (ii)$ actually holds when G has arbitrary real rank [GW, Th. 5.3].

A proof of Lemma 2.3 will be given in Section 2.4. We first recall an interpretation of the homogeneous space G/MA.

2.2. **The homogeneous space G/MA.** When $\mathrm{rank}_{\mathbb{R}}(G) = 1$, there are two G-orbits in $G/P \times G/P$: a closed one, namely

$$\mathrm{Diag}(G/P \times G/P) = \{(\xi, \xi) : \xi \in G/P\},$$

and an open one, namely

$$(G/P)^{(2)} = (G/P \times G/P) \smallsetminus \mathrm{Diag}(G/P \times G/P),$$

which we shall denote by $\mathcal{O}$. If we see G/P as the space of minimal parabolic subgroups of G, with the transitive action of G by conjugation, then $(P, P^-) \in G/P \times G/P$ belongs to $\mathcal{O}$; its stabilizer is $P \cap P^- = MA$, hence $\mathcal{O}$ identifies with G/MA. We can interpret $\mathcal{O}$ as the space of oriented geodesics of $X = G/K$, with the two projections of $\mathcal{O} \subset G/P \times G/P$ onto G/P giving the two endpoints in $\partial_\infty X$. This is known as the *Hopf parameterization* of the space of oriented geodesics of X.

When G has arbitrary real rank, G/MA still identifies with the only open G-orbit in $G/P \times G/P$; there may be more than two G-orbits.

2.3. **Anosov representations.** In this section G can have any real rank.

2.3.1. *Exponential expansion and contraction.* We shall use the following classical terminology. Let $\pi : \mathcal{V} \to \mathcal{M}$ be a vector bundle over a compact manifold $\mathcal{M}$. For any $m \in \mathcal{M}$, choose a norm $\|\cdot\|_m$ on the fiber $\pi^{-1}(m)$ so that the family $(\|\cdot\|_m)_{m \in \mathcal{M}}$ is continuous.

Definition 2.4. A flow $(f_t)_{t\in\mathbb{R}}$ on $\mathcal{V}$ is said to *exponentially expand* (resp. *contract*) $\mathcal{V}$ if there exist $c, C > 0$ such that

$$\|f_t \cdot v\|_{\pi(f_t \cdot v)} \geq C\, e^{c|t|}\, \|v\|_{\pi(v)}$$

for all $t \geq 0$ (resp. $t \leq 0$) and all $v \in \mathcal{V}$.

This does not depend on the choice of the continuous family $(\|\cdot\|_m)_{m\in\mathcal{M}}$, by compactness of $\mathcal{M}$.

2.3.2. *Anosov representations for fundamental groups of closed, negatively curved manifolds.* Let $\mathcal{N}$ be a closed, negatively curved, Riemannian manifold, with universal covering $\widetilde{\mathcal{N}}$ and fundamental group Γ_0. The geodesic flow $(\varphi_t)_{t\in\mathbb{R}}$ on the unit tangent bundle $\mathcal{M} := T^1(\mathcal{N})$ is an *Anosov flow*: there exists a $(d\varphi_t)_{t\in\mathbb{R}}$-invariant splitting

$$T\mathcal{M} = \mathscr{D}^0 \oplus \mathscr{D}^+ \oplus \mathscr{D}^- \tag{2.1}$$

of the tangent bundle $T\mathcal{M}$ into subbundles such that $\mathscr{D}^0$ is tangent to the flow and $\mathscr{D}^+$ (resp. $\mathscr{D}^-$) is exponentially expanded (resp. exponentially contracted) by $(d\varphi_t)_{t\in\mathbb{R}}$ in the sense of Definition 2.4. The notion of Anosov representation aims to generalize this idea to representations of Γ_0 into G. In order to define it, we first make a few remarks. Let $\rho : \Gamma_0 \to G$ be a group homomorphism.

• The group Γ_0 acts on $\widehat{\mathcal{M}} := T^1(\widetilde{\mathcal{N}})$ by deck transformations. We denote by $\widehat{\mathcal{M}} \times_\rho G/MA$ the quotient of $\widehat{\mathcal{M}} \times G/MA$ by the diagonal action

$$\gamma \cdot (\hat{m}, gMA) = \big(\gamma \cdot \hat{m},\, \rho(\gamma)gMA\big)$$

of Γ_0. It is a principal bundle over $\mathcal{M}$ with fiber G/MA.

• The geodesic flow on $\widehat{\mathcal{M}}$, which we also denote by $(\varphi_t)_{t\in\mathbb{R}}$, lifts to a flow $(\psi_t)_{t\in\mathbb{R}}$ on $\widehat{\mathcal{M}} \times G/MA$ which is trivial on the right factor:

$$\psi_t \cdot (\hat{m},\, gMA) = (\varphi_t \cdot \hat{m},\, gMA).$$

Since $(\varphi_t)_{t\in\mathbb{R}}$ commutes with the action of Γ_0 on $\widehat{\mathcal{M}}$, the flow $(\psi_t)_{t\in\mathbb{R}}$ descends to a flow on $\widehat{\mathcal{M}} \times_\rho G/MA$, still denoted $(\psi_t)_{t\in\mathbb{R}}$, which lifts $(\varphi_t)_{t\in\mathbb{R}}$.

• Recall from Section 2.2 that the homogeneous space G/MA identifies with an open subset of $G/P \times G/P$. This product structure induces a G-invariant splitting

$$T(G/MA) = \mathscr{E}^+ \oplus \mathscr{E}^-.$$

In turn, the G-invariant distributions $\mathscr{E}^\pm$ on G/MA determine distributions $\mathscr{F}^\pm \subset T(\widehat{\mathcal{M}} \times_\rho G/MA)$ on $\widehat{\mathcal{M}} \times_\rho G/MA$, which are preserved by the flow $(d\psi_t)_{t\in\mathbb{R}}$.

Definition 2.5 (Labourie [L]). A representation $\rho \in \mathrm{Hom}(\Gamma_0, G)$ is *P-Anosov* if there exists a continuous section s of the bundle $\widehat{\mathcal{M}} \times_\rho G/MA \to \mathcal{M}$ which is compatible with the flows $(\varphi_t)_{t\in\mathbb{R}}$ and $(\psi_t)_{t\in\mathbb{R}}$, in the sense that $s\circ\varphi_t = \psi_t\circ s$ for all $t \in \mathbb{R}$, and such that $s^*\mathscr{F}^+$ (resp. $s^*\mathscr{F}^-$) is exponentially expanded (resp. contracted) by $(d\psi_t)_{t\in\mathbb{R}}$.

Another way to say that s is compatible with the flows $(\varphi_t)_{t\in\mathbb{R}}$ and $(\psi_t)_{t\in\mathbb{R}}$ is to say that it is flat along the flow lines of $(\varphi_t)_{t\in\mathbb{R}}$, for the flat connection on the bundle $\widehat{\mathcal{M}} \times_\rho G/MA \to \mathcal{M}$ induced by the flat connection coming from the product structure on $\widehat{\mathcal{M}} \times G/MA \to \widehat{M}$.

2.3.3. *An equivalent definition.* A section s of the bundle $\widehat{\mathcal{M}}\times_\rho G/MA \to \mathcal{M}$ determines a unique ρ-equivariant map $\sigma : \widehat{\mathcal{M}} \to G/MA$, and vice versa. Therefore, Definition 2.5 can be reformulated as follows: a representation $\rho \in \mathrm{Hom}(\Gamma_0, G)$ is P-Anosov if and only if there exists a continuous, ρ-equivariant map $\sigma : \widehat{\mathcal{M}} \to G/MA$ with the following properties:

- σ is invariant under $(\varphi_t)_{t\in\mathbb{R}}$,
- for some (hence any) continuous, ρ-equivariant family $(\|\cdot\|_{\hat{m}})_{\hat{m}\in\widehat{\mathcal{M}}}$, where $\|\cdot\|_{\hat{m}}$ is a norm on $T_{\sigma(\hat{m})}(G/MA)$, there exist $c, C > 0$ such that

$$\|w\|_{\varphi_t\cdot\hat{m}} \geq C\, e^{c|t|}\, \|w\|_{\hat{m}} \tag{2.2}$$

  for all $\hat{m} \in \widehat{\mathcal{M}}$ and all $(w,t) \in (\mathscr{E}^+_{\sigma(\hat{m})} \times \mathbb{R}^+) \cup (\mathscr{E}^-_{\sigma(\hat{m})} \times \mathbb{R}^-)$.

In this case, σ factors through the space $\widehat{\mathcal{M}}/(\varphi_t)_{t\in\mathbb{R}}$ of oriented geodesics of $\widetilde{\mathcal{N}}$ and is unique: any $\gamma \in \Gamma_0 \smallsetminus \{e\}$ has a unique attracting (resp. repelling) fixed point ξ^+_γ (resp. ξ^-_γ) in G/P, and σ must send the oriented translation axis of γ in $\widetilde{N}$ to $(\xi^+_\gamma, \xi^-_\gamma) \in G/MA$.

2.3.4. *A characterization in terms of roots.* Here is an equivalent formulation of (2.2) due to Guichard–Wienhard [GW], which we give in the special case where $\mathrm{rank}_{\mathbb{R}}(G) = 1$. Since A is contractible, by using a partition of unity we can lift the continuous, ρ-equivariant map $\sigma : \widehat{\mathcal{M}} \to G/MA$ to a continuous, ρ-equivariant map $\sigma' : \widehat{\mathcal{M}} \to G/M$; any two such lifts differ by a bounded amount in any G-invariant Riemannian metric on G/M. Note that A acts on G/M by right multiplication (because M and A commute). For any $\hat{m} \in \widehat{\mathcal{M}}$, there exists a continuous map $t \mapsto a_{\hat{m},t} \in A$ such that

$$\sigma'(\varphi_t \cdot \hat{m}) = \sigma'(\hat{m})\, a_{\hat{m},t}$$

for all $t \in \mathbb{R}$. The tangent space $T(G/M)$ admits a natural right-A-invariant splitting

$$T(G/M) = \mathscr{G}^0 \oplus \mathscr{G}^+ \oplus \mathscr{G}^-$$

into subbundles such that the image of $\mathscr{G}^0$ (resp. $\mathscr{G}^+$, resp. $\mathscr{G}^-$) under the projection $T(G/M) \to T(G/MA)$ is $\{0\}$ (resp. $\mathscr{E}^+$, resp. $\mathscr{E}^-$). To simplify, suppose that the rank-one group G has only one positive restricted root α (in general, when G is not split, it can have two positive roots α and 2α). Then for any $a \in A$, the differential map $T(G/M) \to T(G/M)$ of the right multiplication by a restricts to the trivial map on $\mathscr{G}^0$, to the multiplication by the scalar $\alpha(a) > 0$ on $\mathscr{G}^+$, and to the multiplication by $1/\alpha(a)$ on $\mathscr{G}^-$. From this observation (and its variant when G has two positive roots α and 2α), one easily deduces that (2.2) is equivalent to the existence of $c, C > 0$ such that

$$\alpha(a_{\hat{m},t}) \geq C\, e^{ct} \tag{2.3}$$

for all $(\hat{m},t) \in \widehat{\mathcal{M}} \times \mathbb{R}_+$. Note that if $\hat{m}$ belongs to a periodic orbit of length T of the geodesic flow $(\varphi_t)_{t\in\mathbb{R}}$, corresponding to an element $\gamma \in \Gamma_0$, then

$$\sigma'(\gamma\cdot\hat{m}) = \rho(\gamma)\cdot\sigma'(\hat{m}) = \sigma'(\hat{m})\, a_{\hat{m},T},$$

hence $\rho(\gamma)$ is conjugate in G to an element of $Ma_{\hat{m},T} \subset MA$ and $\log\alpha(a_{\hat{m},T})$ is the length in $\rho(\Gamma)\backslash X$ of the closed orbit corresponding to $\rho(\gamma)$. A formulation of (2.2) similar to (2.3), but involving more than one root α, holds when G has arbitrary real rank (see [GW, § 3.3]).

2.3.5. *Anosov representations for general Gromov-hyperbolic groups.* The definition of Anosov representation has been extended in [GW] to any Gromov-hyperbolic group Γ_0: by [Gr, M], there is a natural action of the group $\Gamma_0 \times \mathbb{R} \rtimes (\mathbb{Z}/2\mathbb{Z})$ on some proper hyperbolic metric space $\widehat{\Gamma}_0$ (the *flow space*, replacing $\widehat{\mathcal{M}} = T^1(\widetilde{\mathcal{N}})$ in the situation above) such that

- the action of $\Gamma_0 \times (\mathbb{Z}/2\mathbb{Z})$ on $\widehat{\Gamma}_0$ is isometric;
- every orbit map $\Gamma_0 \to \widehat{\Gamma}_0$ is a quasi-isometry; in particular, $\partial_\infty\Gamma_0 \simeq \partial_\infty\widehat{\Gamma}_0$;
- the action of $\mathbb{R}$ on $\widehat{\Gamma}_0$ is free, and every orbit map $\mathbb{R} \to \widehat{\Gamma}_0$ is a quasi-isometric embedding; the induced map $\widehat{\Gamma}_0/\mathbb{R} \to (\partial_\infty\widehat{\Gamma}_0)^{(2)}$ is a homeomorphism.

A representation $\rho : \Gamma_0 \to G$ is said to be *P-Anosov* if there exists a continuous, ρ-equivariant, $\mathbb{R}$-invariant map $\sigma : \widehat{\Gamma}_0 \to G/MA$ and, for any continuous, ρ-equivariant family $(\|\cdot\|_{\hat{m}})_{\hat{m}\in\widehat{\Gamma}_0}$ of norms on $T_{\sigma(\hat{m})}(G/MA)$, constants $c, C > 0$ such that (2.2) holds for all $\hat{m} \in \widehat{\Gamma}_0$ and all (w,t) in $\mathscr{E}^+_{\sigma(\hat{m})} \times \mathbb{R}^+$ or $\mathscr{E}^-_{\sigma(\hat{m})} \times \mathbb{R}^-$. As above, this is equivalent to the existence of $c, C > 0$ such that (2.3) holds for all $(\hat{m},t) \in \widehat{\Gamma}_0 \times \mathbb{R}_+$.

2.4. **Proof of Lemma 2.3.** The implication $(i) \Rightarrow (ii)$ is immediate: if Γ is convex cocompact, then it acts cocompactly on $\mathrm{Conv}(\Lambda_\Gamma) \subset X$, hence it is finitely generated and any orbit map $\Gamma \to X$ is a quasi-isometric embedding. On the other hand, any orbit map $G \to X$ is a quasi-isometry since K is compact.

To establish $(ii) \Rightarrow (i)$, it is sufficient to see that if Γ is finitely generated and quasi-isometrically embedded in G, then for any $x \in X$ the set F_Γ of Definition 2.1.(2) is contained in a uniform neighborhood of the Γ-orbit of x. Choose a finite generating subset S of Γ and let

$$r = \max_{s\in S} d_X(x, s\cdot x) > 0.$$

Quasi-isometricity implies the existence of constants $c, C > 0$ such that for any $\gamma, \gamma' \in \Gamma$, there is a (c,C)-quasi-geodesic $\mathcal{G}_{\gamma,\gamma'}$ from $\gamma\cdot x$ to $\gamma'\cdot x$ which is a concatenation of geodesic segments of length $\leq r$ with endpoints in $\Gamma\cdot x$. Since X is Gromov-hyperbolic, there is a constant $R > 0$ (depending only on (c,C) and on the hyperbolicity constant of X) such that any point of the geodesic segment $[\gamma\cdot x, \gamma'\cdot x]$ lies at distance $\leq R$ from $\mathcal{G}_{\gamma,\gamma'}$, hence at distance $\leq R + r$ from $\Gamma\cdot x$. Since Λ_Γ is the set of accumulation points of $\Gamma\cdot x$, we obtain that F_Γ remains at distance $\leq R + r$ from $\Gamma\cdot x$ by taking a limit.

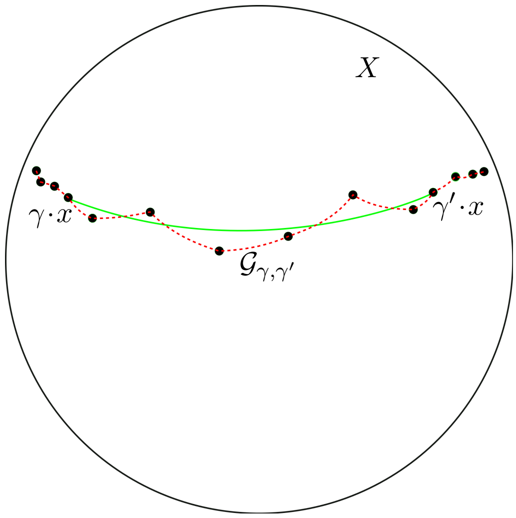


FIGURE 1. If Γ is quasi-isometrically embedded in G, then all geodesics of X with both endpoints in the limit set Λ_Γ are contained in some uniform neighborhood of $\Gamma \cdot x$.

For $(ii) \Rightarrow (iii)$, note that if the natural inclusion of Γ in G is a quasi-isometric embedding, then so is any orbit map $\Gamma \to X$ by compactness of K. In particular, Γ is Gromov-hyperbolic and any orbit map induces a continuous, injective, Γ-equivariant map $\xi : \partial_\infty \Gamma \to \partial_\infty X = G/P$.

To establish $(iii) \Rightarrow (iv)$, we can use the natural homeomorphisms $\widehat{\Gamma}/\mathbb{R} \simeq (\partial_\infty \Gamma)^{(2)}$ and $(G/P)^{(2)} \simeq G/MA$ from Sections 2.2 and 2.3 to obtain, from ξ, a continuous, Γ-equivariant, $\mathbb{R}$-invariant map

$$\sigma : \widehat{\Gamma} \longrightarrow G/MA.$$

Let $\sigma' : \widehat{\Gamma} \to G/M$ be a continuous, Γ-equivariant lift of σ. By Section 2.3, it is sufficient to prove the existence of constants $c, C > 0$ such that (2.3) holds for all $\hat{m} \in \widehat{\Gamma}$ and $t \geq 0$. This follows from the compactness of $\Gamma \backslash \widehat{\Gamma}$: see [GW, Th. 4.11] for details in the case that Γ is Zariski-dense in G. (We can reduce to the Zariski-dense case by using the general fact [Bo] that a Zariski-closed subgroup of G is either reductive or contained in a proper parabolic subgroup of G.)

The implication $(iv) \Rightarrow (ii)$ is true when G has arbitrary real rank; we explain it for $\mathrm{rank}_{\mathbb{R}}(G) = 1$. (The general proof is similar, using a version of (2.3) involving more than one root α.) Recall from Section 2.3 that if (iv) holds, then there exist a continuous, Γ-equivariant, $\mathbb{R}$-invariant map $\sigma : \widehat{\Gamma} \to G/MA$ and, for any continuous, Γ-equivariant lift $\sigma' : \widehat{\Gamma} \to G/M$ of σ, constants $c, C > 0$ such that (2.3) holds for all $\hat{m} \in \widehat{\Gamma}$ and $t \geq 0$. The condition (2.3) implies that the restriction of σ' to any periodic $\mathbb{R}$-orbit is a (c, C')-quasi-isometric embedding, for some $C' > 0$ independent of the $\mathbb{R}$-orbit. On the other hand, the hyperbolicity of $\widehat{\Gamma}$ implies the existence of a constant $R > 0$ with the following property: for any $y, y' \in \widehat{\Gamma}$, there exist $z, z' \in \widehat{\Gamma}$ such that z and z' belong to the same $\mathbb{R}$-orbit and the distances in $\widehat{\Gamma}$ between y and z, and between y' and z', are $\leq R$. Therefore, $\sigma' : \widehat{\Gamma} \to G/M$

is a quasi-isometric embedding. Since M is compact, Γ is quasi-isometrically embedded in G.

3. Generalizations of convex cocompactness to higher rank

We now allow the semisimple group G to have arbitrary real rank. Let G_1 (resp. G_2) be the product of the simple factors of G of real rank 1 (resp. of real rank ≥ 2). (Products are in fact almost products, but this is not very important.) The Riemannian symmetric space $X = G/K$ is the product of the Riemannian symmetric spaces X_1 of G_1 and X_2 of G_2.

Fix a discrete subgroup Γ of G. If H is the Zariski-closure of Γ in G and $R(H)$ its radical, then the group $H/R(H)$ is semisimple and the image of Γ in $H/R(H)$ is discrete (see [R, Cor. 8.27]). We may therefore restrict to the case where Γ is Zariski-dense in G, up to replacing G with a smaller group.

3.1. Generalization of (1).

In answer to a question of Corlette from 1994, Kleiner and Leeb proved that in higher real rank, Definition 2.1.(1) for convex cocompactness does not give any interesting example of discrete groups apart from products of uniform lattices and convex cocompact subgroups of rank-one factors.

Theorem 3.1 (Kleiner–Leeb [KL2])**.** *Let $\mathcal{C} \neq \emptyset$ be a Γ-invariant, closed, convex subset of X with $\Gamma\backslash\mathcal{C}$ compact. If Γ is Zariski-dense in G, then $\mathcal{C} = \mathcal{C}_1 \times X_2$ for some Γ-invariant, closed, convex subset $\mathcal{C}_1$ of X_1; in particular, Γ is a product of convex cocompact subgroups of the rank-1 factors of G_1 and of a uniform lattice of G_2.*

The proof easily reduces to the case that G is simple of real rank ≥ 2. In this case, Kleiner and Leeb use results from [Be] and [KL1] to prove that the geometric boundary of $\mathcal{C}$ is a top-dimensional subbuilding of the spherical building at infinity (Tits boundary) of X, which is a closed subset with respect to the topology of the visual boundary of X. The main step in [KL2] is then to show that any such subbuilding is either equal to the full visual boundary of X or contained in the visual boundary of a proper symmetric subspace of X. This last possibility is ruled out by the Zariski-density assumption on Γ, and so $\mathcal{C} = X$.

On the other hand, Quint [Q] investigated generalizations of the definitions (2) and (3) of convex cocompactness from Section 2.1. Before we explain his work, we recall interpretations of the homogeneous spaces G/P, G/MA, and G/M in arbitrary real rank.

3.2. The homogeneous spaces G/P, G/MA, and G/M.

In real rank 1, the homogeneous space G/P identifies with the visual boundary of $X = G/K$, namely with the set of equivalence classes of geodesic rays of X for the relation "to be at finite Hausdorff distance". In higher real rank, the visual boundary of X is more complicated; its regular part contains infinitely many copies of G/P. However, we can see G/P as the *Furstenberg boundary* of X, which is the set of equivalence classes of *Weyl chambers* of X for the relation "to be at finite Hausdorff distance".

Recall that a *maximal flat* of X is a flat, totally geodesic subspace of X that is maximal for inclusion. For instance, the A-orbit of the origin $x_0 :=$

$eK \in G/K = X$ is a maximal flat $\mathcal{F}_0$, and the other maximal flats are the G-translates of $\mathcal{F}_0$; they are all isometric to $\mathbb{R}^n$ where $n = \mathrm{rank}_{\mathbb{R}}(G) = \dim A$. Let $\overline{A^+}$ be the closed positive Weyl chamber of A corresponding to our choice of positive restricted root system: by definition, $\overline{A^+}$ is the subset of A on which all positive restricted roots take values ≥ 1. A *Weyl chamber* of X is a G-translate of $\overline{A^+} \cdot x_0 \subset X$. The Weyl group

$$W := N_G(A)/Z_G(A) = N_G(A)/MA$$

(where $N_G(A)$ denotes the normalizer of A in G) naturally acts on A; this induces a faithful action of W on $\mathcal{F}_0 = A \cdot x_0$, which is generated by the orthogonal reflections along some hyperplanes $\mathcal{H}_1, \dots, \mathcal{H}_n$ of $\mathcal{F}_0$ bounding the Weyl chamber $\overline{A^+} \cdot x_0$.

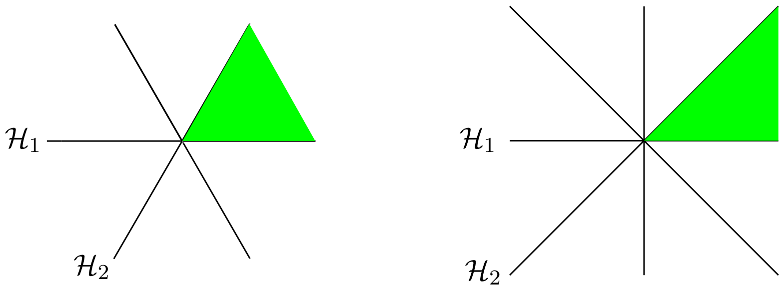


FIGURE 2. Weyl chambers in a maximal flat of X for $G = \mathrm{SL}_3(\mathbb{R})$ (type A_2) and for $G = \mathrm{Sp}_4(\mathbb{R})$ (type C_2).

As in Section 2.2, the homogeneous space G/MA identifies with the unique open G-orbit $\mathcal{O}$ in $G/P \times G/P$. (In higher rank there are more than two G-orbits.) Pairs of elements of G/P that belong to $\mathcal{O}$ are said to be *in general position*.

Example 3.2. For $G = \mathrm{SL}_{n+1}(\mathbb{R})$, the space G/P identifies with the set of maximal flags $(V_1 \subset \cdots \subset V_n)$ of $\mathbb{R}^{n+1}$, and the open G-orbit $\mathcal{O}$ is the set of pairs of maximal flags

$$(\xi, \eta) = \big((V_1 \subset \cdots \subset V_n), (V_1' \subset \cdots \subset V_n')\big)$$

such that $V_i \oplus V'_{n+1-i} = \mathbb{R}^{n+1}$ for all $1 \leq i \leq n$. For any $(\xi, \eta) \in \mathcal{O}$ we can find a basis $(e_1, \dots, e_{n+1})$ of $\mathbb{R}^{n+1}$ that is *adapted* to (ξ, η), in the sense that

$$\begin{aligned} \xi &= \big(\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, \dots, e_n \rangle\big), \\ \eta &= \big(\langle e_{n+1} \rangle \subset \langle e_n, e_{n+1} \rangle \subset \cdots \subset \langle e_2, \dots, e_{n+1} \rangle\big). \end{aligned}$$

• We shall identify G/P with the set of equivalence classes of Weyl chambers of X by mapping $gP \in G/P$ to $g \cdot \xi_0^+$ for all $g \in G$, where ξ_0^+ is the class of the Weyl chamber $\overline{A^+} \cdot x_0 \subset \mathcal{F}_0$. (The stabilizer of ξ_0^+ in G is P.)

• We shall identify G/MA with the open G-orbit $\mathcal{O} \subset G/P \times G/P$ by mapping $gMA \in G/MA$ to $g \cdot (\xi_0^+, \xi_0^-)$ for all $g \in G$, where ξ_0^- is the class of the Weyl chamber $(\overline{A^+})^{-1} \cdot x_0 \subset \mathcal{F}_0$. (The element ξ_0^- has stabilizer P^- in G, hence (ξ_0^+, ξ_0^-) has stabilizer $P \cap P^- = MA$.)

• The set G/MA surjects G-equivariantly onto the set of maximal flats of X by mapping (ξ_0^+, ξ_0^-) to $\mathcal{F}_0$. The fiber is the Weyl group W, acting on $\mathcal{F}_0 = A \cdot x_0$ as above.

$$\begin{array}{ccccc} G/MA & \xrightarrow{\sim} & \mathcal{O} & \xrightarrow{\text{fiber } W} & \{\text{maximal flats of } X\} \\ gMA & \longmapsto & g \cdot (\xi_0^+, \xi_0^-) & \longmapsto & g \cdot \mathcal{F}_0 \end{array} \tag{3.1}$$

We shall also consider the homogeneous space G/M, which identifies with the actual set of Weyl chambers of X (*not* modulo equivalence) with the natural action of G. When $\mathrm{rank}_{\mathbb{R}}(G) = 1$, a Weyl chamber is a geodesic ray, hence is determined by its endpoint in X and its direction at the endpoint: in this case, G/M identifies with the unit tangent bundle $T^1(X)$. When $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$, the space $T^1(X)$ has larger dimension than G/M:

$$\begin{aligned} \dim(T^1(X)) &= 2\dim(X) - 1 = 2\dim(\mathfrak{a}) + 2\dim(\mathfrak{n}) - 1, \\ \dim(G/M) &= \dim(G) - \dim(M) = \dim(\mathfrak{a}) + 2\dim(\mathfrak{n}), \end{aligned}$$

where the right-hand equality in the first (resp. second) line follows from the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ (resp. from (1.1)). As in Section 2.3, the group A acts on G/M by right multiplication. When $\mathrm{rank}_{\mathbb{R}}(G) = 1$ (i.e. $\dim(A) = 1$), this is the geodesic flow on $T^1(X)$. In general, the right action of A on G/M (or on $\Gamma\backslash G/M$) is called the *Weyl chamber flow.*

3.3. **Generalization of (2) and (3).** Suppose that Γ is Zariski-dense in G. By work of Benoist [Be], when $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$ there still exists (as in the classical situation where $\mathrm{rank}_{\mathbb{R}}(G) = 1$) a smallest, nonempty, closed, Γ-invariant subset Λ_Γ of G/P, called the *limit set* of Γ. Explicitly, Λ_Γ is the set of points $\xi \in G/P$ such that some sequence $(\gamma_n)_{n\in\mathbb{N}}$ of elements of Γ "contracts G/P towards ξ", in the sense that the push-forward by γ_n of the K-invariant probability measure ν on $G/P \simeq K/M$ converges weakly to the Dirac mass in ξ (i.e. $\nu(\gamma_n^{-1} \cdot \mathcal{U}) \to 1$ for any open subset $\mathcal{U}$ of G/P containing ξ). Limit sets in higher real rank were first introduced by Guivarc'h [Gh] for $G = \mathrm{SL}_{n+1}(\mathbb{R})$.

Let A^+ be the interior of $\overline{A^+}$ in A (open positive Weyl chamber). An element $g \in G$ is said to be *proximal* if it is conjugate to an element of MA^+. In this case, it has a unique attracting fixed point ξ_g^+ and a unique repelling fixed point ξ_g^- in G/P; these two points are in general position.

Benoist proved that the limit set Λ_Γ is in fact the closure in G/P of the set of fixed points ξ_γ^+ for proximal $\gamma \in \Gamma$ [Be, Lem. 3.6.(iii)]. Moreover, the set $(\Lambda_\Gamma \times \Lambda_\Gamma) \cap G/MA$ is the closure in G/MA of the set of pairs $(\xi_\gamma^+, \xi_\gamma^-)$ for proximal $\gamma \in \Gamma$ [Be, Lem. 3.6.(iv)].

We shall consider the following generalization of Definition 2.1.(2) for convex cocompactness:

(2) Γ acts cocompactly on F_Γ,

where we define F_Γ be the union of all maximal flats of X which are images of points of $(\Lambda_\Gamma \times \Lambda_\Gamma) \cap G/MA$ under (3.1). We shall also consider the following generalization of Definition 2.1.(3) for convex cocompactness:

(3) the intersection of the $\mathfrak{a}^+$-conservative and $(-\mathfrak{a}^+)$-conservative sets of the Weyl chamber flow on $\Gamma\backslash G/M$ is compact.

Here we set $\mathfrak{a}^+ = \log A^+ \subset \mathfrak{a}$ and use the following terminology (where $\|\cdot\|$ is any fixed norm on $\mathfrak{a}$).

Definition 3.3. Let Ω be an open cone of $\mathfrak{a}$. A point $x \in \Gamma\backslash G/M$ is Ω-*conservative* if there exists a sequence $(a_n)_{n\in\mathbb{N}}$ of elements of A, tending to infinity, such that $(xa_n)_{n\in\mathbb{N}}$ is bounded in $\Gamma\backslash G/M$ and $\log(a_n)/\|\log(a_n)\|$ converges to a point of Ω. The Ω-*conservative set* of the Weyl chamber flow on $\Gamma\backslash G/M$ is the closure of the set of Ω-conservative points.

Quint established the following relations between the generalized definitions (1), (2), (3) of convex cocompactness.

Lemma 3.4 (Quint [Q]). *In any real rank,*

$$(1) \Longrightarrow (2) \Longleftrightarrow (3).$$

He also proved that in higher real rank, the definitions (2) and (3) for convex cocompactness do not give any interesting example of discrete groups apart from products of uniform lattices and convex cocompact subgroups of rank-one factors.

Theorem 3.5 (Quint [Q]). *If Γ is Zariski-dense and satisfies (2) or (3), then Γ is a product of convex cocompact subgroups of the rank-1 factors of G_1 and of a uniform lattice of G_2; in particular, any nonempty, Γ-invariant, closed, convex subset $\mathcal{C}$ of X is of the form $\mathcal{C} = \mathcal{C}_1 \times X_2$ for some Γ-invariant closed convex subset $\mathcal{C}_1$ of X_1.*

In the rest of these notes, we shall explain the essential ideas of the proof of Lemma 3.4 and Theorem 3.5.

4. Main steps in the proof of Theorem 3.5

4.1. **Proof of** (2) $\Leftrightarrow$ (3)**.** Let E'_Γ be the (full) preimage of $(\Lambda_\Gamma \cap \Lambda_\Gamma) \cap G/MA$ in G/M, and let $E_\Gamma = \Gamma\backslash E'_\Gamma \in \Gamma\backslash G/M$. In other words,

$$E_\Gamma = \left\{ \Gamma g M \in \Gamma\backslash G/M : \ g \cdot (\xi_0^+, \xi_0^-) \in \Lambda_\Gamma \times \Lambda_\Gamma \right\} . \tag{4.1}$$

By construction, E_Γ is closed and right-A-invariant. Note that the image of $E'_\Gamma \subset G/M$ in $X = G/K$ is F_Γ. Since the fiber K/M is compact, the compactness of E_Γ is equivalent to the cocompactness of the action of Γ on F_Γ.

$$\begin{tikzcd}
& E'_\Gamma & E_\Gamma \\
& \cap & \cap \\
& G/M \arrow[r] \arrow[dl] \arrow[dr] & \Gamma\backslash G/M \\
G/MA & & X = G/K \\
\cup & & \cup \\
(\Lambda_\Gamma \cap \Lambda_\Gamma) \cap G/MA & & F_\Gamma
\end{tikzcd}$$

The equivalence (2) $\Leftrightarrow$ (3) in Lemma 3.4 is an immediate consequence of this observation and of the following lemma.

Lemma 4.1 [Q, Cor. 4.2]**.** *In any real rank, E_Γ is the intersection of the $\mathfrak{a}^+$-conservative and $(-\mathfrak{a}^+)$-conservative sets of the Weyl chamber flow on $\Gamma\backslash G/M$.*

Proof. Consider $x = \Gamma g M \in \Gamma\backslash G/M$.

If x is $\mathfrak{a}^+$-conservative, then by definition there exist a sequence $(a_n) \in A^{\mathbb{N}}$ going to infinity, a bounded sequence $(\kappa_n) \in G^{\mathbb{N}}$, and a sequence $(\gamma_n) \in \Gamma^{\mathbb{N}}$, such that $\log(a_n)/\|\log(a_n)\|$ converges to a point of $\mathfrak{a}^+$ and

$$g\, a_n = \gamma_n\, \kappa_n$$

for all $n \in \mathbb{N}$. Up to passing to a subsequence, we may assume that $\kappa_n \to \kappa$ for some $\kappa \in G$. The subset $\mathcal{U}_0$ of G/P consisting of points that are in general position with ξ_0^- is open and dense in G/P. By Remark 4.2 below, for any $\xi \in G/P$ with $\kappa^{-1} \cdot \xi \in \mathcal{U}_0$,

$$\gamma_n \cdot \xi = g a_n \kappa_n^{-1} \cdot \xi \underset{n\to+\infty}{\longrightarrow} g \cdot \xi_0^+,$$

and this convergence is uniform on compact sets. Therefore, if ν is the K-invariant probability measure on G/P, we have $\nu(\gamma_n^{-1} \cdot \mathcal{U}) \to 1$ for any open subset $\mathcal{U}$ of G/P containing $g \cdot \xi_0^+$, which means by definition that $g \cdot \xi_0^+ \in \Lambda_\Gamma$. Similarly, if x is $(-\mathfrak{a}^+)$-conservative, then $g \cdot \xi_0^- \in \Lambda_\Gamma$, hence $x \in E_\Gamma$.

Conversely, suppose $x \in E_\Gamma$, i.e. $g \cdot (\xi_0^+, \xi_0^-) \in \Lambda_\Gamma \times \Lambda_\Gamma$. If $g \cdot (\xi_0^+, \xi_0^-) = (\xi_\gamma^+, \xi_\gamma^-)$ for some loxodromic element $\gamma \in \Gamma$, then $g^{-1}\gamma g \in Ma$ for some $a \in A^+$. We have $xa^n = x = xa^{-n}$ for all n, hence x is both $\mathfrak{a}^+$-conservative and $(-\mathfrak{a}^+)$-conservative. In general, we use the density [Be, Lem. 3.6.(iv)] of the set of pairs $(\xi_\gamma^+, \xi_\gamma^-)$ in $(\Lambda_\Gamma \times \Lambda_\Gamma) \cap G/MA$. ☐

In the proof we have used the following remark.

Remark 4.2. The subset $\mathcal{U}_0$ of G/P consisting of points that are in general position with ξ_0^- is the P^--orbit of ξ_0^+. It identifies with $P^-/(P \cap P^-)$, or equivalently with $\mathfrak{n}^-$ endowed with an action of P^- whose restriction to MA is the adjoint action ($\xi_0^+ \in \mathcal{U}_0$ corresponds to $0 \in \mathfrak{n}^-$). In particular, for any sequence $(a_n) \in A^{\mathbb{N}}$, if $\alpha(a_n) \to +\infty$ for all positive restricted roots α, then $a_n \cdot \xi \to \xi_0^+$ for all $\xi \in \mathcal{U}_0$, and the convergence is uniform on compact sets.

4.2. **Strategy of proof of Theorem 3.5.** Note that the Weyl group $W = N_G(A)/MA$ acts on G/MA by multiplication on the right.

Suppose that (2) and (3) are satisfied. Then the right-A-invariant set E_Γ is compact, hence any point of E_Γ is Ω-conservative for any open cone Ω of $\mathfrak{a}$. Taking $\Omega = w \cdot \mathfrak{a}^+$ for w ranging over W, applying Lemma 4.1, and using (4.1), we obtain that Λ_Γ is *W-stable*, in the following sense.

Definition 4.3. A subset Λ of G/P is *W-stable* if $(\Lambda \times \Lambda) \cap G/MA$ is right-W-invariant, which means that for any $g \in G$ such that $g \cdot (\xi_0^+, \xi_0^-) \in \Lambda \times \Lambda$ we have $gw \cdot \xi_0^+ \in \Lambda$ for all $w \in W$.

Theorem 3.5 therefore reduces to the following proposition.

Proposition 4.4 [Q, Prop. 3.1]**.** *The only closed, Zariski-dense, W-stable subset Λ of G/P is G/P.*

Quint first establishes Proposition 4.4 in the case that G is simple of real rank 2, i.e. for restricted root systems of type A_2 (e.g. $G = \mathrm{SL}_3(\mathbb{R})$), of type $B_2 = C_2$ (e.g. $G = \mathrm{Sp}_4(\mathbb{R})$), and of type G_2. This relies on a combinatorial argument, which we give in Section 5 for $G = \mathrm{SL}_3(\mathbb{R})$. He then reduces the general case to the rank-2 case, as we explain for $G = \mathrm{SL}_n(\mathbb{R})$ in Section 6.

5. Proof of Proposition 4.4 for $G = \mathrm{SL}_3(\mathbb{R})$

In this section we take $G = \mathrm{SL}_3(\mathbb{R})$ and see G/P as the set of pairs (p, ℓ) where p is a point of $\mathbb{RP}^2$ and $\ell \subset \mathbb{RP}^2$ a projective line containing p.

Let Λ be a closed, Zariski-dense, W-stable subset of G/P. We denote by Λ^0 (resp. Λ^1) the projection of Λ to the set of points (resp. projective lines) of $\mathbb{RP}^2$; it is Zariski-dense in $\mathbb{RP}^2$ (resp. in the space of projective lines of $\mathbb{RP}^2$) since Λ is Zariski-dense in G/P.

For any projective line $\ell \subset \mathbb{RP}^2$ and any point $q \in \mathbb{RP}^2 \smallsetminus \ell$, we call *projection from q onto ℓ* the map from $\mathbb{RP}^2 \smallsetminus \{q\}$ to ℓ that sends any projective line through q (minus $\{q\}$) to its intersection point with ℓ.

5.1. Preliminary remarks.

5.1. Preliminary remarks. Before we prove Proposition 4.4, we make a few useful observations leading to a reduction of the proposition.

(i) The W-stability of Λ means that the six flags determined by any pair of elements of Λ all belong to Λ (see Figure 3).

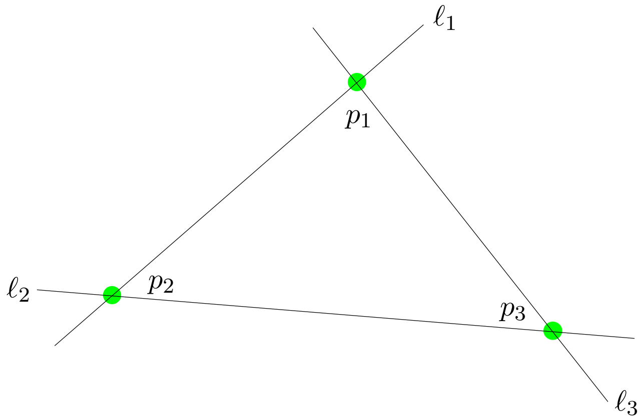


Figure 3. If (p_1, ℓ_1) and (p_2, ℓ_2) belong to the W-stable set Λ, then so do (p_1, ℓ_3), (p_2, ℓ_1), (p_3, ℓ_2), and (p_3, ℓ_3).

In particular, for any $\ell_1 \neq \ell_2$ in Λ^1 the intersection $\ell_1 \cap \ell_2$ belongs to Λ^0, for any $p_1 \neq p_2$ in Λ^0 the projective line through p_1 and p_2 belongs to Λ^1, and the projection from a point $q \in \Lambda^0$ onto a line $\ell \in \Lambda^1$ disjoint from q preserves Λ^0.

(ii) This implies that for any $\ell \in \Lambda^1$ the set $\ell \cap \Lambda^0$ is Zariski-dense in ℓ, and for any $p \in \Lambda^0$ the set of projective lines of Λ^1 through p is Zariski-dense in the set of projective lines of $\mathbb{RP}^2$ through p.

Indeed, for the first statement, fix $\ell \in \Lambda^1$ and choose an arbitrary point $q \in \Lambda^0 \smallsetminus \ell$. By (i), the projection from q onto ℓ maps the Zariski-dense subset $\Lambda^0 \smallsetminus \{q\}$ of $\mathbb{RP}^2$ to $\Lambda^0 \cap \ell$. Therefore, $\Lambda^0 \cap \ell$ is Zariski-dense in ℓ. The second statement is similar.

(iii) In order to establish Proposition 4.4 for $G = \mathrm{SL}_3(\mathbb{R})$, it is sufficient to prove the following.

Claim 5.1. *Any line $\ell \in \Lambda^1$ is contained in Λ^0, and any projective line through a point $p \in \Lambda^0$ belongs to Λ^1.*

Indeed, consider $(p,\ell) \in G/P$. Since Λ is Zariski-dense in G/P, we can find an element $(p',\ell') \in \Lambda$ in general position with (p,ℓ). Claim 5.1 gives successively that $(\ell \cap \ell') \in \Lambda^0$, that $\ell \in \Lambda^1$, and that $p \in \Lambda^0$. In other words, there exist a projective line ℓ_1 through p and a point $p_2 \in \ell$ such that (p,ℓ_1) and (p_2,ℓ) belong to Λ. Then $(p,\ell) \in \Lambda$ by (i).

5.2. **Reduction to a rank-1 result.** In order to prove Claim 5.1, we can reduce to real rank 1 and use the following lemma.

Lemma 5.2 [Q, Prop. 2.1]**.** *Suppose G has real rank 1 and let Λ be a Zariski-dense subset of G/P with the following property: for any $\xi \in \Lambda$ the group of unipotent elements $u \in G$ such that $u \cdot \xi = \xi$ and $u \cdot \Lambda = \Lambda$ acts transitively on $\Lambda \smallsetminus \{\xi\}$. Then $\Lambda = G/P$.*

Indeed, suppose that Lemma 5.2 is proved and consider $\ell \in \Lambda^1$ and $p \in \ell$. By Lemma 5.2, in order to prove that $p \in \Lambda^0$, it is sufficient to see that the set of projective transformations u of ℓ that preserve $\Lambda^0 \cap \ell$ and admit p as a unique fixed point, acts transitively on $(\Lambda^0 \cap \ell) \smallsetminus \{p\}$. (Such transformations correspond to unipotent elements of the stabilizer of ℓ, which is isomorphic to $\mathrm{SL}_2(\mathbb{R})$.) To see this, consider $p_1 \neq p_2$ in $(\Lambda^0 \cap \ell) \smallsetminus \{p\}$.
- By (ii), there exist two distinct lines $\ell_1, \ell_2 \in \Lambda^1$ through p, different from ℓ.
- By (ii) again, there exist $p_3 \neq p$ in $\ell_1 \cap \Lambda^0$. Let us denote by p_4 (resp. p_5) the image of p_1 (resp. p_2) under the projection from p_3 onto ℓ_2 (see Figure 4).
- By (i), the points p_4 and p_5 belong to Λ^0.
- By (i) again, the projection $\mathrm{proj}_{p_4,\ell_1}$ from p_4 onto ℓ_1 preserves Λ^0, and so does the projection $\mathrm{proj}_{p_5,\ell}$ from p_5 onto ℓ. In particular, the map

$$u := \mathrm{proj}_{p_5,\ell} \circ \mathrm{proj}_{p_4,\ell_1},$$

which preserves ℓ, also preserves $\Lambda^0 \cap \ell$. It maps p_1 to p_2 and admits p as a unique fixed point. This completes the proof of Claim 5.1, hence of Proposition 4.4 for $G = \mathrm{SL}_3(\mathbb{R})$.

5.3. **Proof of Lemma 5.2.** We may assume that G is connected. Let S be the stabilizer of Λ in G; it is a closed subgroup of G.
- S is Zariski-dense in G. Indeed, let H be the identity component (for the real topology) of the Zariski closure of G. The flag variety of H is a Zariski-closed subset of G/P; by assumption, it contains the Zariski-dense set Λ, hence it is equal to G/P. This implies $H = G$.
- S is not discrete in G. Indeed, since Λ is infinite and G/P is compact, Λ admits a nonisolated point η: there exists a sequence $(\eta_n)_{n\in\mathbb{N}}$ of pairwise distinct points of Λ converging to η. By assumption, for any n there exists a unipotent element $u_n \in S$ such that $u_n \cdot \xi = \xi$ and $u_n \cdot \eta = \eta_n$. The sequence $(u_n)_{n\in\mathbb{N}}$ converges to 1, hence S is not discrete.

This implies that $S = G$ (a Zariski-dense subgroup of a connected semisimple Lie group is either discrete or dense), and so $\Lambda = G/P$.

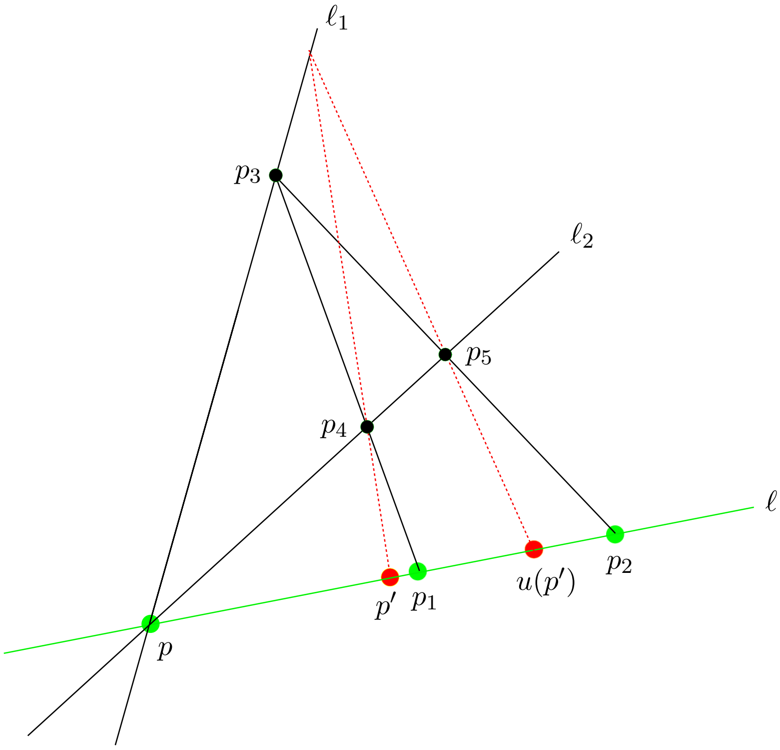


Figure 4. Construction of a unipotent transformation u.

6. Reducing to rank-2 groups in Proposition 4.4

To simplify notation and make the proof more concrete, we suppose that $G = \mathrm{SL}_{n+1}(\mathbb{R})$, that A is the subgroup of diagonal matrices with positive entries, and that P is the group of upper triangular matrices (Example 1.1). The space G/P identifies with the set of maximal flags

$$(V_1 \subset \cdots \subset V_n)$$

of $\mathbb{R}^{n+1}$ (Example 3.2). Let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, for $1 \leq i \leq n$, be the standard simple roots of A in G. For any i, let P_i be the parabolic subgroup of G associated with $\{\alpha_i\}$, consisting of upper triangular block matrices with square diagonal blocks of size $1, \ldots, 1, 2, 1, \ldots, 1$, where 2 is the i-th entry. For $\xi = g \cdot \xi_0^+ \in G/P$, we set

$$P_i^{\xi} = gP_ig^{-1}.$$

Let Λ be a closed, Zariski-dense, W-stable subset of G/P. To show that $\Lambda = G/P$, we first note that for any $\xi, \eta \in G/P$ in general position, there is a finite sequence

$$\xi = \xi_1,\ \xi_2,\ \ldots,\ \xi_{k+1} = \eta$$

of elements of G/P such that for any $1 \leq j \leq k$ we have $\xi_{j+1} \in P_{i_j}^{\xi_j} \cdot \xi_j$ for some $1 \leq i_j \leq n$. Indeed, write the longest element

$$w_0 : (1, \ldots, n+1) \longmapsto (n+1, \ldots, 1)$$

of the Weyl group $W \simeq \mathfrak{S}_{n+1}$ as a product $\tau_1 \dots \tau_k$ of transpositions $\tau_j : (i_j, i_j + 1) \mapsto (i_j + 1, i_j)$; if $(\xi, \eta) = g \cdot (\xi_0^+, \xi_0^-)$, then we may take

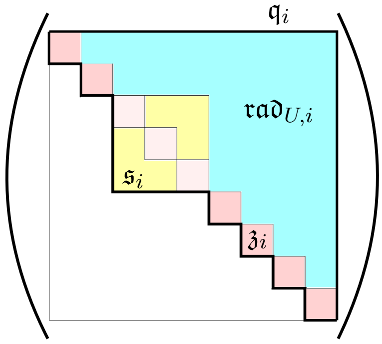
$$\xi_{j+1} := g\tau_1 \dots \tau_j \cdot \xi_0^+$$

for all j. By induction, it is sufficient to prove that if $\xi_j \in \Lambda$, then $\xi_{j+1} \in \Lambda$.

Let us prove a stronger statement, namely that for any $\xi \in \Lambda$ and any $1 \leq i \leq n-1$ we have $Q_i^\xi \cdot \xi \subset \Lambda$, where Q_i^ξ is the parabolic subgroup of G generated by P_i^ξ and P_{i+1}^ξ. To simplify notation, we assume that $\xi = \xi_0^+$ (the proof for general ξ is similar). The group $Q_i := Q_i^{\xi_0^+}$ admits a Levi decomposition

$$Q_i = Z_i \, S_i \ltimes \mathrm{Rad}_{U,i},$$

where $\mathrm{Rad}_{U,i} \subset N$ is the unipotent radical of Q_i (i.e. the largest connected normal unipotent subgroup), $Z_i \subset A$ is the center of Q_i (consisting of diagonal matrices whose i-th, $(i+1)$-th, and $(i+2)$-th entries are equal), and S_i is a semisimple Lie group (isomorphic to $\mathrm{SL}_3(\mathbb{R})$). The orbit $Q_i \cdot \xi_0^+$ identifies with $Q_i/(P \cap Q_i)$ and, seen as an S_i-homogeneous space, with $S_i/(P \cap S_i)$, which is the flag variety of $S_i \simeq \mathrm{SL}_3(\mathbb{R})$.



In order to prove that $Q_i \cdot \xi_0^+ \subset \Lambda$, assuming $\xi_0^+ \in \Lambda$, it is enough (by the rank-2 case of Proposition 4.4 treated in Section 5) to prove the following.

Claim 6.1. *The set $\Lambda \cap (Q_i \cdot \xi_0^+)$, seen as a subset of the flag variety of S_i, is closed, Zariski-dense, and stable under the Weyl group of S_i.*

Proof. Let $\mathrm{pr}_1 : G/MA \simeq \mathcal{O} \subset G/P \times G/P \to G/P$ be the projection onto the first factor and let $w_i \in W \simeq \mathfrak{S}_{n+1}$ be the transposition $(i, i+2) \mapsto (i+2, i)$. Recall that the Weyl group $W = N_G(A)/MA$ acts on G/MA on the right. For any $\xi \in G/P$, let $\mathcal{U}_\xi$ be the (Zariski-dense) set of elements of G/P that are in general position with ξ. We define a map $\pi_i^\xi : \mathcal{U}_\xi \to G/P$ by

$$\pi_i^\xi(\eta) = \mathrm{pr}_1\big((\xi, \eta) \cdot w_i\big) \tag{6.1}$$

for all $\eta \in \mathcal{U}_\xi$. Concretely, if $(e_1, \dots, e_{n+1})$ is a basis of $\mathbb{R}^{n+1}$ adapted to (ξ, η) (see Example 3.2), then

$$\pi_i^\xi(\eta) = \big(V_1 \subset \dots \subset V_{i-1} \subset V_i' \subset V_{i+1}' \subset V_{i+2} \subset \dots \subset V_n\big) \tag{6.2}$$

where $\xi = (V_1 \subset \dots \subset V_n)$ and

$$V_i' := V_{i-1} + \mathbb{R}\, e_{i+2}, \qquad V_{i+1}' := V_{i-1} + \mathbb{R}\, e_{i+1} + \mathbb{R}\, e_{i+2}. \tag{6.3}$$

We now restrict to $\xi \in Q_i \cdot \xi_0^+$. From (6.2) and (6.3) we see that the set $\pi_i^\xi(\mathcal{U}_\xi)$ is contained in $Q_i^\xi \cdot \xi = Q_i \cdot \xi_0^+$, and in fact identifies with the set of elements of $S_i/(P \cap S_i)$ that are in general position with the image of ξ; in particular, $\pi_i^\xi(\mathcal{U}_\xi)$ is Zariski-dense in $Q_i \cdot \xi_0^+ \simeq S_i/(P \cap S_i)$. From (6.1) we see that:

- π_i^ξ is an algebraic homomorphism; in particular, since $\Lambda \cap \mathcal{U}_\xi$ is Zariski-dense in $\mathcal{U}_\xi$, the set $\pi_i^\xi(\Lambda \cap \mathcal{U}_\xi)$ is Zariski-dense in $\pi_i^\xi(\mathcal{U}_\xi)$, hence in $Q_i \cdot \xi_0^+$;
- if $\xi \in \Lambda$, then $\pi_i^\xi(\Lambda \cap \mathcal{U}_\xi) \subset \Lambda$ since Λ is W-stable.

Therefore, the set $\Lambda \cap (Q_i \cdot \xi_0^+)$ is Zariski-dense in $Q_i \cdot \xi_0^+$. Let us prove that it is W_i-stable as a subset of $S_i/(P \cap S_i)$, where

$$W_i = N_{S_i}(A \cap S_i)/Z_{S_i}(A \cap S_i)$$

is the Weyl group of S_i, which embeds in W as the subgroup generated by the transpositions $(i, i+1) \mapsto (i+1, i)$ and $(i+1, i+2) \mapsto (i+2, i+1)$. From (6.1) and the W-stability of Λ we see that for any $\xi \in \Lambda \cap (Q_i \cdot \xi_0^+)$ and any $\eta \in \Lambda \cap \mathcal{U}_\xi$ we have $(\xi, \pi_i^\xi(\eta)) \cdot w_i' \in \Lambda \times \Lambda$ for all $w_i' \in W_i$. Therefore, it is sufficient to prove that for any pair (ξ, ζ) of points of $\Lambda \cap (Q_i \cdot \xi_0^+)$ that are in general position in $Q_i \cdot \xi_0^+ \simeq S_i/(P \cap S_i)$, we can write $\zeta = \pi_i^\xi(\eta)$ for some $\eta \in \Lambda \cap \mathcal{U}_\xi$. Fix such a pair (ξ, ζ) and consider a point $\tau \in \Lambda \cap \mathcal{U}_\xi \cap \mathcal{U}_\zeta$; then we may take $\eta := \pi_{n-i}^\tau(\zeta)$. $\square$

7. Proof of (1) ⇒ (2) in Lemma 3.4

In order to prove that (1) implies (2), it is sufficient to check the following.

Lemma 7.1 [Q, Lem. 5.4]**.** *The set F_Γ is contained in any nonempty, Γ-invariant, closed, convex subset $\mathcal{C}$ of $X = G/K$.*

Recall from Section 3.3 that F_Γ is the set of maximal flats of X corresponding to elements of $(\Lambda_\Gamma \times \Lambda_\Gamma) \cap G/MA$ under (3.1). For $(\xi, \eta) \in G/MA$, we shall denote by $\mathcal{F}(\xi, \eta)$ the corresponding maximal flat of X.

Recall that any element of G admits a *Jordan decomposition*: $g = g_h g_e g_u$ for some unique commuting elements $g_h, g_e, g_u \in G$ with g_h hyperbolic (i.e. conjugate to an element $a_g \in \overline{A^+}$), with g_e elliptic (i.e. conjugate to an element of M), and with g_u unipotent. We denote by $\lambda : G \to \overline{\mathfrak{a}^+} := \log \overline{A^+}$ the *Lyapunov projection* of G, defined by $\lambda(g) = \log a_g$ for all g. By definition, the *limit cone* $\mathcal{L}_\Gamma$ is the smallest closed cone of $\overline{\mathfrak{a}^+}$ containing $\lambda(\Gamma)$. Benoist [Be] proved that if Γ is Zariski-dense in G, then $\mathcal{L}_\Gamma$ is convex with nonempty interior.

Sketch of proof of Lemma 7.1. The set of pairs $(\xi_\gamma^+, \xi_\gamma^-)$ for loxodromic $\gamma \in \Gamma$ is dense in $(\Lambda_\Gamma \times \Lambda_\Gamma) \cap G/MA$ [Be, Lem. 3.6.(iv)]. Therefore, by convexity of $\mathcal{C}$, it is sufficient to prove that $\mathcal{F}(\xi_\gamma^+, \xi_\gamma^-) \subset \mathcal{C}$ for any loxodromic $\gamma \in \Gamma$. This is done in four steps:

(a) The flat $\mathcal{F}(\xi_\gamma^+, \xi_\gamma^-)$ meets $\mathcal{C}$. Indeed, for any point $x \in \mathcal{C}$, the geodesic segments $[\gamma^n \cdot x, \gamma^{-n} \cdot x]$, for $n \in \mathbb{N}$, are contained in $\mathcal{C}$ and converge (up to passing to a subsequence) to a geodesic line in $\mathcal{F}(\xi_\gamma^+, \xi_\gamma^-) \cap \mathcal{C}$ (see [Q, Lem. 5.2]).

(b) By [Be, Lem. 4.2 bis], for any $v \in \mathcal{L}_\Gamma$ with $\|v\| = 1$ there is a sequence $(\gamma_n)_{n\in\mathbb{N}}$ of loxodromic elements of Γ such that $\lambda(\gamma_n)/\|\lambda(\gamma_n)\| \to v$ and $(\xi^+_{\gamma_n}, \xi^-_{\gamma_n}) \to (\xi^+_\gamma, \xi^-_\gamma)$.

(c) This implies that for any $v \in \mathcal{L}_\Gamma$ with $\|v\| = 1$, the set $\mathcal{F}(\xi^+_\gamma, \xi^-_\gamma) \cap \mathcal{C}$ contains a geodesic of direction v (after identification of $\mathcal{F}(\xi^+_\gamma, \xi^-_\gamma)$ with $\mathcal{F}_0$, hence with $\mathfrak{a}$, which is well-defined up to the action of the Weyl group W). Indeed, fix $x \in \mathcal{F}(\xi^+_\gamma, \xi^-_\gamma) \cap \mathcal{C}$. For any $n \in \mathbb{N}$, there is a geodesic of direction $\lambda(\gamma_n)$ in $\mathcal{F}(\xi^+_{\gamma_n}, \xi^-_{\gamma_n}) \cap \mathcal{C}$ whose distance to x is bounded independently of n (see [Q, Lem. 5.2]). Up to passing to a subsequence, these lines converge to a geodesic of direction v in $\mathcal{F}(\xi^+_\gamma, \xi^-_\gamma) \cap \mathcal{C}$.

(d) The limit cone $\mathcal{L}_\Gamma$ has nonempty interior [Be, Th. 1.2]. By (c), the set $\mathcal{F}(\xi^+_\gamma, \xi^-_\gamma) \cap \mathcal{C}$ contains lines whose directions form a cone with nonempty interior. This implies $\mathcal{F}(\xi^+_\gamma, \xi^-_\gamma) \subset \mathcal{C}$ by convexity of $\mathcal{C}$. □

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