LECTURE NOTES FOR CONVEX REAL PROJECTIVE STRUCTURES

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Introduction

This short note consists of two parts.

Part I is to introduce the convex $\mathbb{R}P^2$-structures on a compact surface and to explain that the deformation space of such structures on a closed surface S of genus $g > 1$ form a moduli space $\mathcal{B}(S)$ homeomorphic to an open cell of dimension $16(g-1)$. This work is done by Choi and Goldman.

Part II is to explain the correspondence between the deformation space $\mathcal{B}(S)$ and the space of pairs (Σ, U), where Σ is a Riemann surface varying in Teichmüller space and U is a cubic differential on Σ. Teichmüller space $\mathcal{T}(S)$ embeds inside $\mathcal{B}(S)$ as the locus of pairs $(\Sigma, 0)$, where Σ is a Riemann surface varying in Teichmüller space. This work is independently done by Labourie and Loftin.

Part I: Deformation space of convex $\mathbb{R}P^2$-structures

1. Convex $\mathbb{R}P^2$-structures

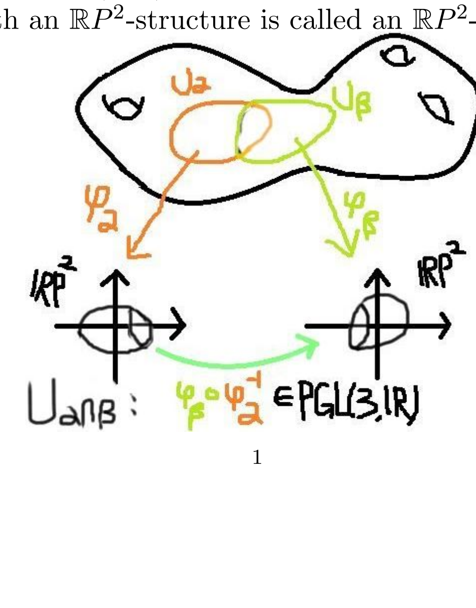
Definition 1. An $\mathbb{R}P^2$-structure on a smooth 2-manifold M is a system of coordinate charts $\{(U_\alpha, \phi_\alpha)\}$ satisfying

(i) $\{U_\alpha\}$ is an open cover of M;

(ii) $\psi_\alpha : U_\alpha \to \mathbb{R}P^2$ is a diffeomorphism onto its image;

(iii) $\psi_\beta \circ {\psi_\alpha}^{-1} \in PGL(3, \mathbb{R})$.

A manifold with an $\mathbb{R}P^2$-structure is called an $\mathbb{R}P^2$-maniflod.



Definition 2. An $\mathbb{R}P^2$-structure on M is called convex if its developing map is a diffeomorphism of $\widetilde{M}$ onto a convex domain Ω in some affine $\mathbb{R}^2 \subset \mathbb{R}P^2$. In this case, we can realize $M = \Omega/\Gamma$, where Γ is a subgroup of $PGL(3, \mathbb{R})$ which acts discretely and properly discontinuous on Ω.

Fact. *Let $M = \Omega/\Gamma$ be a closed surface with a convex $\mathbb{R}P^2$-structure. Suppose that $\chi(M) < 0$. Then the following hold.*
(1) $\Omega \subset \mathbb{R}P^2$ is a strictly convex domain with C^1 boundary and therefore contains no affine line.
(2) Either $\partial\Omega$ is a conic in $\mathbb{R}P^2$ or in not $C^{1+\epsilon}$ for some $0 < \epsilon < 1$.
(3) The attracting and repelling fixed points of elements of Γ form a dense subset of $\partial\Omega$. Furthermore given any pair $(x, y) \in \partial\Omega \times \partial\Omega$, there exists a sequence $\gamma_n \in \Gamma$ such that $Fix_+(\gamma_n) \to x$, and $Fix_-(\gamma_n) \to y$.

Let S denote a compact surface.

Definition 3. $\mathbb{R}P^2(S) := \{(f, M) | f : S \to M$ is a diffeomorphism and M is an $\mathbb{R}P^2$-manifold$\}/$, where (f, M) (f', M') if there exists a projective isomorphism $h : M \to M'$ such that $h \circ f$ is isotopic to f'.

$\mathcal{B}(S)$ denotes the subset of $\mathbb{R}P^2(S)$ corresponding to convex $\mathbb{R}P^2$-structures.

Fact. *(1) The set of equivalent classes $\mathbb{R}P^2(S)$ have a natural topology making it locally equivalent to $Hom(\pi, PGL(3, \mathbb{R}))/PGL(3, \mathbb{R})$, i.e., there exists a holonomy map $hol : \mathbb{R}P^2(S) \to Hom(\pi, PGL(3, \mathbb{R}))/PGL(3, \mathbb{R})$ which is a local diffeomorphism.*
(2) $\mathcal{B}(S)$ is open in $\mathbb{R}P^2(S)$.

Excercise 1. *Show the following.*
(1) $\mathbb{R}P^2(S)$ is a Hausdorff real analytic manifold of dimension $-8\chi(S)$;
(2) The restriction of $hol : \mathbb{R}P^2(S) \to Hom(\pi, SL(3, \mathbb{R}))/SL(3, \mathbb{R})$ to $\mathcal{B}(S)$ is an embedding of $\mathcal{B}(S)$ onto a Hausdorff real analytic manifold of dimension $-8\chi(S)$.

Our main goal to show the following theorem.

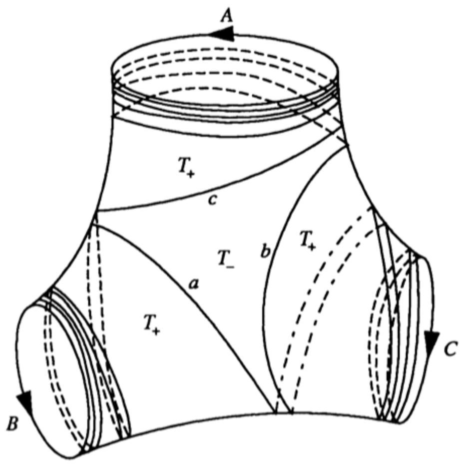
MainTheorem. *(Goldman [3]) Let S be a compact surface having a boundary components such that $\chi(S) < 0$. Then $\mathcal{B}(S)$ is diffeomorphic to a celll of dimension $-8\chi(S)$ and the map which associates to a convex $\mathbb{R}P^2$-structures neaat ∂M is a fibration of $\mathcal{B}(S)$ over an open $2n$-cell with fiber an open cell of dimension $-8\chi(S) - 2n$.*

Corollary 1.1. *Let S be a closed orientable surface of genus $g > 1$. Then $\mathcal{B}(S)$ is diffeomorphic to an open celll of dimension $16(g - 1)$.*

To prove the above main theorem, we cut a surface into pair-of-pants as what we did in Teichmueller theory. We first try to understand the simplest case when where S is a pair-of-pants. Then if we know how to assemble the convex $\mathbb{R}P^2$-structures on pair-of-pants together, especially if the convex property is preserved (which is indeed true), we are almost done.

2. Convex $\mathbb{R}P^2$-structures on a pair-of-pants

In this section, we prove Theorem 1 in the case where S is a pair-of-pants with A, B, C three components.

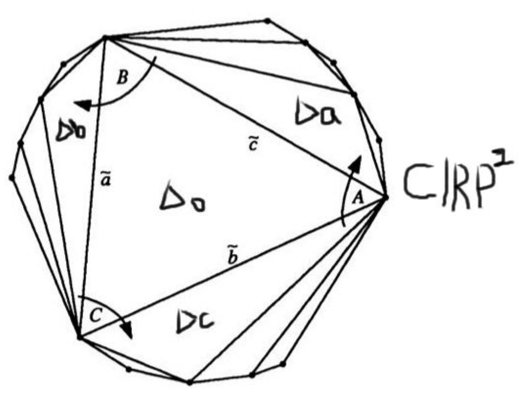


Theorem 2.1. *The deformation space $\mathcal{B}(S)$ of convex $\mathbb{R}P^2$-structures on S is an open 8-dimension cell.*

Sketch of Proof.

Suppose that $A \in SL(3, \mathbb{R})$. We define $\lambda(A)$ to be the real eigenvalue of $A \in SL(3, \mathbb{R})$ having the smallest absolute value and $\tau(A) \in \mathbb{R}$ as the sum of the other two (possibly unreal) eigenvalues. We can show that the pair $(\lambda(A), \tau(A))$ is a complete invariant of the $SL(3, \mathbb{R})$-conjugacy class of the boundary A. Such pairs form a 2-cell.

Denote $\mathcal{O}$ as the space of $SL(3, \mathbb{R})$-conjugation classes of the set of all $(\triangle_0, \triangle_a, \triangle_b, \triangle_c, A, B, C)$ satisfying conditions described in following picture.



$CBA = I$

the union $\overline{\Delta}_0 \cup \overline{\Delta}_a \cup \overline{\Delta}_b \cup \overline{\Delta}_c$ is a convex hexagon;

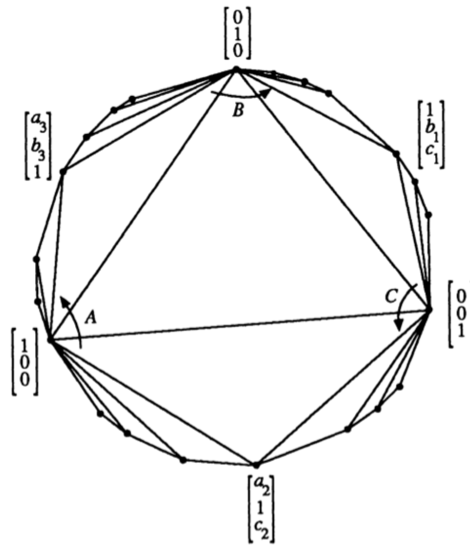
The reason we consider $\mathcal{O}$ instead of $\mathcal{B}(S)$ is because:

(1) $\mathcal{O}$ is computable and,

(2) Each $\mathbb{R}P^2$-structure corresponding to a point in $\mathcal{O}$ is convex, i.e.,$\mathcal{O} = \mathcal{B}(S)$. (the claim needs more proof.)

Now it is sufficient to show the map $\theta_{\partial S} : \mathcal{B}(S) \to \mathcal{B}(\partial S)$ obtained by associating to a convex structure the boundary invariants $((\lambda, \tau)_A, (\lambda, \tau)_B, (\lambda, \tau)_C)$ is a fibration over an open 6-cell with fiber a 2-dimensional open cell.

Since $\mathcal{O}$ considers the conjugation class under $SL(3, \mathbb{R})$, then we can consider the set $(\triangle_0, \triangle_a, \triangle_b, \triangle_c)$ as following picture parametrized by $(b_1, c_1, a_2, b_2, a_3, b_3)$. Note this set is not fixed under diagonal matrices $\mathrm{diag}(\lambda, \mu, \nu)$ with $\lambda\mu\nu = 1$. So actually we have 4 dimensional parameters for the conjugation clases $(\triangle_0, \triangle_a, \triangle_b, \triangle_c)$.



Here A is parametrized by new parameters $(\alpha_1, \beta_1, \gamma_1)$, written as

$$\begin{aligned} A &= \alpha_1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot [1 \quad a_2 \quad 0] + \beta_1 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot [0 \quad -1 \quad 0] \\ &+ \gamma_1 \cdot \begin{bmatrix} a_3 \\ b_3 \\ -1 \end{bmatrix} \cdot [0 \quad c_2 \quad 1] \\ &= \begin{bmatrix} \alpha_1 & \alpha_1 a_2 + \gamma_1 a_3 c_1 & \gamma_1 a_3 \\ 0 & -\beta_1 + \gamma_1 b_3 c_2 & \gamma_1 b_3 \\ 0 & -\gamma_1 c_2 & -\gamma_1 \end{bmatrix}, \end{aligned}$$

Here B is parametrized by new parameters $(\alpha_2, \beta_2, \gamma_2)$, written as

$$
\begin{aligned}
B &= \alpha_2 \cdot \begin{bmatrix} -1 \\ b_1 \\ c_1 \end{bmatrix} \cdot [1 \quad 0 \quad a_3] + \beta_2 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot [0 \quad 1 \quad b_3] \\
&\quad + \gamma_2 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot [0 \quad 0 \quad -1] \\
&= \begin{bmatrix} -\alpha_2 & 0 & -\alpha_2 a_3 \\ \alpha_2 b_1 & \beta_2 & \beta_2 b_3 + \alpha_2 a_3 b_1 \\ \alpha_2 c_1 & 0 & -\gamma_2 + \alpha_2 a_3 c_1 \end{bmatrix},
\end{aligned}
$$

Here C is parametrized by new parameters $(\alpha_3, \beta_3, \gamma_3)$, written as

$$
\begin{aligned}
C &= \alpha_3 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot [-1 \quad 0 \quad 0] + \beta_3 \cdot \begin{bmatrix} a_2 \\ -1 \\ c_2 \end{bmatrix} \cdot [b_1 \quad 1 \quad 0] \\
&\quad + \gamma_3 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot [c_1 \quad 0 \quad 1] \\
&= \begin{bmatrix} -\alpha_3 + \beta_3 a_2 b_1 & \beta_3 a_2 & 0 \\ -\beta_3 b_1 & -\beta_3 & 0 \\ \gamma_3 c_1 + \beta_3 b_1 c_2 & \beta_3 c_2 & \gamma_3 \end{bmatrix},
\end{aligned}
$$

From $\det(A) = \det(B) = \det(C) = 1$, we obtain that

$$\alpha_1 \beta_1 \gamma_1 = \alpha_2 \beta_2 \gamma_2 = \alpha_3 \beta_3 \gamma_3 = 1$$

From $CBA = 1$, we obtain that

$$\alpha_1 \alpha_2 \alpha_3 = \beta_1 \beta_2 \beta_3 = \gamma_1 \gamma_2 \gamma_3 = 1.$$

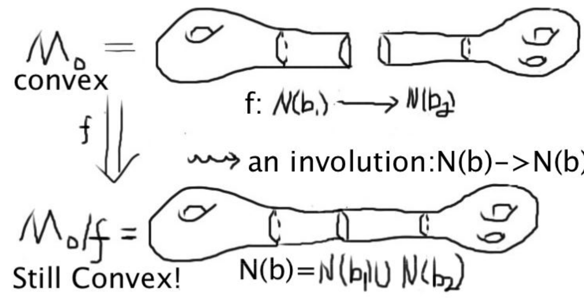
The above six equations actually can be reduced to 5 restrictions. Hence considering (A, B, C) gives us 9-5=4 more parameters. Hence we can see $\dim(\mathcal{O}) = 4 + 4 = 8$.

We are not finished since we need to know $\mathcal{O}$ is a cell. Actually, if we are careful, we may be able to choose two more right parameters (s, t) valued in $\mathbb{R}_+$ besides the boundary invariants $((\lambda, \tau)_A, (\lambda, \tau)_B, (\lambda, \tau)_C)$, then we are done.

3. Assembling convex $\mathbb{R}P^2$-structures

The proof is based on the Fenchel-Nilsen coordinate system on Teichmueller space. The proof of Main Theorem is based on the following two lemmas,

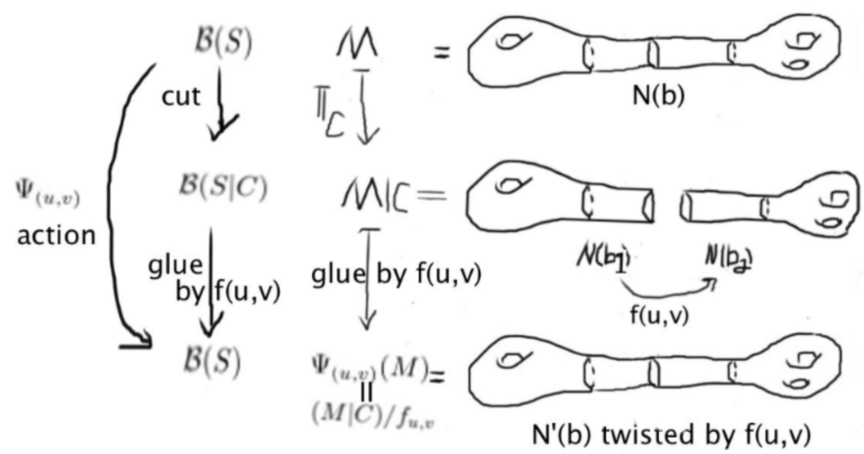
Lemma 3.1. (***Gluing Convex*** $\mathbb{R}P^2$***-manifolds***) *Let M_0 be a compact convex $\mathbb{R}P^2$-manifold with principal boundary, and suppose that $b_1, b_2 \subset \partial M_0$ are boundary components with collar neighborhoods $b_i \subset N(b_i) \subset M_0 (i = 1, 2)$. Suppose that $f : N(b_1) \to N(b_2)$ is a projective isomorphism. Then the $\mathbb{R}P^2$-manifold M_0/f obtained by identification by $f : N(b_1) \to N(b_2)$ is a* ***convex*** *$\mathbb{R}P^2$-manifold.*



From the above lemma, we know that gluing convex $\mathbb{R}P^2$-structures along a geodesic curve C only depends on the projective isomorphism f on the neighborhood of C. Then conversely we actually know how much information we lose if we cut a convex $\mathbb{R}P^2$-structure along a geodesic curve, which is stated in the following lemma.

Lemma 3.2. (***Gut Convex*** $\mathbb{R}P^2$***-manifolds***) *Suppose that $C \subset S$ is a two-sided simple closed curve such that each component of $S|C$ has negative Euler characteristic. Let $\Pi_C : \mathcal{B}(S) \to$ a subspace of $\mathcal{B}(S|C)$ be the map which arises from splitting a convex $\mathbb{R}P^2$-structure on S along the closed geodesic homotopic to C. Then Π_C is a fibration and each fiber of is an open 2-cell.*

Sketch of Lemma: Let $b_1, b_2 \subset \partial(S|C)$ be the two boundary components corresponding to C. We now define an $\mathbb{R}^2$-action Ψ on $\mathcal{B}(S)$ such that Π_C is a Ψ-invariant fibration onto the subspace of $\mathcal{B}(S|C)$ defined by the conditions $(\lambda, \tau)_{b_1} = (\lambda, \tau)_{b_2}$ and Ψ is simply transitive on each fiber of Π_C. For $(u, v) \in \mathbb{R}^2$ we construct a new convex $\mathbb{R}P^2$-manifold $\Psi_{(u,v)}(M)$ representing a point in $\mathbb{R}P^2$-manifold.
Consider the split $\mathbb{R}P^2$-manifold $M|C$; let $b_1, b_2 \subset \partial(M|C)$ be the two boundary components corresponding to C. For any $(u, v) \in \mathbb{R}^2$, there exists a principal collar neighborhoods $N(b_i) \subset M|C$ of b_i for $i = 1, 2$



Let $T^u = \begin{pmatrix} e^{-u} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^u \end{pmatrix}, U^v = \begin{pmatrix} e^{-v} & 0 & 0 \\ 0 & e^{2v} & 0 \\ 0 & 0 & e^{-v} \end{pmatrix}$, where $u, v \in \mathbb{R}$. Let $f(u,v) : N(b_1) \to N(b_2)$ be a projective isomorphism such that $f(u,v)$ is induced by $T^u U^v$ on the developing image in $\mathbb{R}P^2$. As in Gluing Lemma, there is a corresponding convex $\mathbb{R}P^2$-manifold $(M|C)/f(u,v)$ representing a point $\Psi_{(u,v)}(M)$ in $\mathcal{B}(S)$. This action generalizes the Fenchel-Nilsen twist flows on the Teichmueller space.

MainTheorem. *(Goldman [3]) Let S be a compact surface having a boundary components such that $\chi(S) < 0$. Then $\mathcal{B}(S)$ is diffeomorphic to a celll of dimension $-8\chi(S)$ and the map which associates to a convex $\mathbb{R}P^2$-structures neaat ∂M is a fibration of $\mathcal{B}(S)$ over an open $2n$-cell with fiber an open cell of dimension $-8\chi(S) - 2n$.*

4. Hitchin Component

Hitchin shows that $Hom(\pi, SL(3,\mathbb{R}))/SL(3,\mathbb{R})$ has exactly three connected components:
1. $\mathcal{C}_0$, the component containing the class of the trivial representation;
2. $\mathcal{C}_1$, the component consisting of classes of representatons which do not lift to the double covering of $PGL(3,\mathbb{R})$;
3. $\mathcal{C}_2$, the component containing faithful representations into $SO(2,1)$.

In particaular, the component $\mathcal{C}_2$ contais the Teichmueller space of S. From our exercise, we know the holonomy map **hol** maps $\mathcal{B}(S)$ bijectively onto an open subset in $\mathcal{C}_2$. Furthermore, Hitchin shows that $\mathcal{C}_2$ is homeomorphic to $\mathbb{R}^{16(g-1)}$. This naturally let Hitchin to conjecture that $\mathcal{C}_2 = \mathcal{B}(S)$.

Theorem 4.1. *(Choi and Goldman [2]) $\mathcal{B}(S)$ is closed in $\mathcal{C}_2$. Hence $\mathcal{C}_2 = \mathcal{B}(S)$.*

Part II: Correspondence of $\mathcal{B}(S)$ and spaces of pairs (Σ, U)

From now on, let S and M be a closed orientable surface of genus g. Our goal is to show the following theorem.

MainTheorem. *(Loftin [5], Labourie [4]) There exists a natural bijective correspondence betwen convex $\mathbb{R}P^2$-structures on S and pairs (Σ, U), where Σ is a Riemann surface homeomorphic to S, and U is a holomorphic cubic differential on Σ.*

Benoist and Hulin in [1] generalize the above result to the case when S is not necessarily closed stated as follows.

Theorem 4.2. *(Theorem 1.1 in [1]) Let S be an oriented surface with non-abelian fundamental group. The map $\mathcal{A} \to (J, U)$ is a bijection between:*
1. the set of properly convex projective structures $\mathcal{A}$ on s with finite Finster volume;
2. the set of hyperbolic Riemann surface structures J on S with finite volume

together with a holomorphic cubic differential U on (S, J) with poles of order at most 2 at the cusps.

5. Hyperbolic Affine Sphere

This section is mainly copied from Loftin [5].
Consider a hypersurface immersion $f : H \to \mathbb{R}^3$, and consider a transversal vector field ξ on the hypersurface H. We have the equations:

$$\begin{aligned} D_X f_*(Y) &= f_*(\nabla_X Y) + h(X,Y)\xi \\ D_X \xi &= -f_*(SX) + \beta(X)\xi. \end{aligned}$$

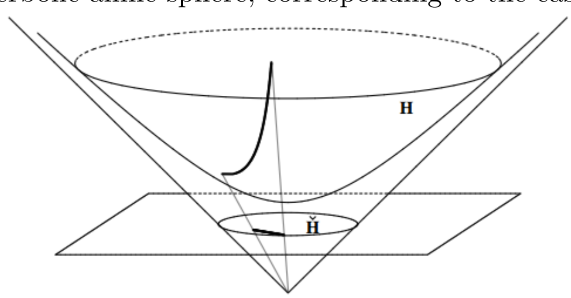
Here, X and Y are tangent vectors on H, the operator D is the canonical flat connection induced from $\mathbb{R}^3$, the operator ∇ is a torsion-free connection, the form h is a symmetric bilinear form on $T_x(H)$, the map S is an endormorphism of $T_x(M)$, and β is a one-form.

For the case the hypersurface H is strictly convex and ξ is the affine normal (see definiton in page 8 in [5]), the structure equations for H become

$$\begin{aligned} D_X f_*(Y) &= f_*(\nabla_X Y) + g(X,Y)\xi \\ D_X \xi &= -f_*(SX). \end{aligned}$$

The connection ∇ is callled the Blaschke connection. The bilinear form h is called the affine metric (since H is strictly convex)and the endormorphism S is called the affine shape operator.

Definition 4. An affine sphere is a hypersurface H in $\mathbb{R}^3$ all of whose affine normals point toward a given point in $\mathbb{R}P^2$, the center of the affine sphere, in this case $S = LI$, where the affine mean curvature L is a constant function on H and I is the identity map. If the center lies on the concave side of H, it is called a hyperbolic affine sphere, corresponding to the case L is negative.



Thus by scaling, we can normalize any hyperbolic affine sphere to have $L = -1$. Also, we can translate so that the center is 0. Then the affine normal $\xi = f$, where f is the embedding of H into $\mathbb{R}^3$. The structure equations then become

$$\begin{aligned} D_X f_*(Y) &= f_*(\nabla_X Y) + g(X,Y)f \\ D_X f &= f_*(X). \end{aligned}$$

Proposition 5.1. *Consider a convex, bounded domain $\Omega \subset \mathbb{R}^2$, where $\mathbb{R}^2$ is embedded in $\mathbb{R}^3$ as the affine space $x_3 = 1$. Then, there is a unique properly embedded hyperbolic affne sphere $H \subset \mathbb{R}^3$ of affine mean curvatuer -1 and center 0 asymptotic to the boundary of the cone $\mathcal{C}(\Omega) \subset \mathbb{R}^3$.*

Proposition 5.2. *Each convex $\mathbb{R}P^2$-structure on M corresponds to an affine sphere structure on M(i.e., M can be realized as a quotient of a byperbolic affine sphere with center 0 and affine mean curvature -1 in the cone $\mathcal{C}$.) vice versa.*

Now we make use of the affine sphere structures to get the correspondence.

6. How to get a holomorphic cubic differential

This section is mainly copied from Loftin [6].
To get a holomorphic cubic differential from this construction, we begin to work with complexified tangent vectors, and we extend ∇, h and D by complex linearity. Consider a local conformal coordinate $z = x + iy$ on the hyperbolic affine sphere with center the origin and affine curvature -1. Then the affine metric is given by $g = e^{\psi}|dz|^2$ for some function ψ. Parametrize the surface by

$$f : \mathcal{D} \to \mathbb{R}^3, \tag{6.1}$$

with $\mathcal{D}$ a domain in $\mathbb{C}$. Then we have the following structure equations for the affine sphere:

$$\begin{aligned} D_X Y &= \nabla_X Y + g(X, Y) f \\ D_X f &= X \end{aligned}$$

Here D is the canonical flat connection on $\mathbb{R}^3$, ∇ is a projectively flat connection, and g is the affine metric. Consider the complexified frame for the tangent bundle to the surface given by $\{e_1 = f_z = f_*(\frac{\partial}{\partial z}), e_{\bar{1}} = f_{\bar{z}} = f_*(\frac{\partial}{\partial \bar{z}})\}$.

Then we have

$$g(f_z, f_z) = g(f_{\bar{z}}, f_{\bar{z}}) = 0,\ g(f_z, f_{\bar{z}}) = \frac{1}{2} e^{\psi}.$$

Consider $\widehat{\theta}, \theta$ the matrix of connection one-forms for the Levi-Civita connection, Blaschke connection respectively. By the above equation,

$$\widehat{\theta}^1_{\bar{1}} = \widehat{\theta}^{\bar{1}}_1 = 0,\ \widehat{\theta}^1_1 = \partial\psi,\ \widehat{\theta}^{\bar{1}}_{\bar{1}} = \bar{\partial}\psi.$$

The difference $\widehat{\theta} - \theta$ is given by the Pick form J. We have

$$\widehat{\theta}^j_i - \theta^j_i = C^j_{ik}\rho^k,$$

where $\{\rho^1 = dz, \rho^{\bar{1}} = d\bar{z}\}$ is the dual frame of one-forms. With the property of the affine normal ξ such that $det(f_z, f_{\bar{z}}, \xi) = \frac{1}{2} i e^{\phi}$ and the totally symmetry of C_{ijk} in affine differential geometry (where $C_{ijk} = C^l_{ij} g_{lk}$, i.e., lower index), this determines θ:

$$\begin{pmatrix} \theta_1^1 & \theta_{\bar{1}}^1 \\ \theta_1^{\bar{1}} & \theta_{\bar{1}}^{\bar{1}} \end{pmatrix} = \begin{pmatrix} \partial\psi & C_{\bar{1}\bar{1}}^1 d\bar{z} \\ C_{11}^{\bar{1}} dz & \bar{\partial}\psi \end{pmatrix} = \begin{pmatrix} \partial\psi & \overline{U}e^{-\psi} d\bar{z} \\ Ue^{-\psi} dz & \bar{\partial}\psi \end{pmatrix},$$

where we define $U = C_{11}^{\bar{1}} e^{\psi}$.

Recall that D is the canonical flat connection induced from $\mathbb{R}^3$. We get the structure equations

$$\begin{aligned} f_{zz} &= \psi_z f_z + Ue^{-\psi} f_{\bar{z}} \\ f_{\bar{z}\bar{z}} &= \overline{U}e^{-\psi} f_z + \psi_{\bar{z}} f_{\bar{z}} \\ f_{z\bar{z}} &= \frac{1}{2} e^{\psi} f \end{aligned}$$

Then, these three equations form a linear first-order system of partial differential equations in $f, f_z, f_{\bar{z}}$, where we may write as

$$\frac{\partial}{\partial z} \begin{pmatrix} f \\ f_z \\ f_{\bar{z}} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \psi_z & Ue^{-\psi} \\ \frac{1}{2}e^{\psi} & 0 & 0 \end{pmatrix} \begin{pmatrix} f \\ f_z \\ f_{\bar{z}} \end{pmatrix},$$

$$\frac{\partial}{\partial \bar{z}} \begin{pmatrix} f \\ f_z \\ f_{\bar{z}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2}e^{\psi} & 0 & 0 \\ 0 & \overline{U}e^{-\psi} & \psi_{\bar{z}} \end{pmatrix} \begin{pmatrix} f \\ f_z \\ f_{\bar{z}} \end{pmatrix}.$$

In order to have a solution of the system, the only condition is that mixed partials must commute (by the Frobenius theorem). Thus we require the two conditions

$$\begin{aligned} \psi_{zz} + U\overline{U}e^{-2\psi} - \frac{1}{2}e^{\psi} &= 0, \\ U_{\bar{z}} &= 0. \end{aligned}$$

By the above argument, we actually construct a map from the space of convex $\mathbb{R}P^2$-structures on S to the space of pairs (Σ, U). The converse direction is mainly by showing the existence and uniqueness of solutions to the above elliptic equation for a given pair (Σ, U). Combining these two directions together, we obtain the following main theorem.

MainTheorem. *(Loftin* [5]*, Labourie* [4]*) Let S be a compact oriented surface of genus $g > 1$. There exists a natural bijective correspondence betwen convex $\mathbb{R}P^2$-structures on S and pairs (Σ, U), where Σ is a Riemann surface homeomorphic to S, and U is a holomorphic cubic differential on Σ.*

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