INTRODUCTION TO POSITIVE REPRESENTATIONS (FOLLOWING FOCK-GONCHAROV)

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Abstract. In this talk, we will try to give a simple description of the set of positive representations of the fundamental group of a surface with non-empty boundary to the group $\mathrm{PSL}(m, \mathbb{R})$, as defined by Fock and Goncharov [1]. For each step of the construction, we will consider the classical cases $m = 2$ and $m = 3$ before turning to the general case. We will also give the main property of these representations, namely their faithfulness and discreteness.

1. Introduction

Given a surface $S_{g,s}$ of genus g with s boundary components, there are well-known coordinates on the (classical) Teichmüller space which are given by Thurston shearing coordinates. On the other hand, points in Teichmuller spaces can also be seen as conjugacy classes of representations of the fundamental group of the surface into the group $\mathrm{PSL}(2, \mathbb{R})$ satisfying various properties (discrete, faithful, etc ...). In this talk, we will show that it is possible to define complex coordinates on the bigger moduli space of representations into $\mathrm{PGL}(2, \mathbb{C})$ (in fact a finite ramified cover over this space) and see that Teichmuller space will be exactly the representations for which the coordinates are real positive. What is more interesting is that it is possible to generalize this picture to the case of $\mathrm{PGL}(m, \mathbb{C})$.

So the purpose of this talk will be to define the set of framed representations of a surface group into $\mathrm{PGL}(m, \mathbb{C})$ and define global coordinates on it using a triangulation of the surface. The coordinates will depend on the triangulation, but the set of representations with positive coordinates will be the same for all triangulations. This set of *positive representations* will be the so-called higher Teichmüller space in the sense of Fock and Goncharov. While the positive representations might be defined for a general split semi-simple real Lie group, using Lustzig notion of total positivity, we will restrict ourselves to the simpler case $G = \mathrm{PSL}(m, \mathbb{R})$ where everything can be made explicit.

To understand the properties of positive representations, we will see how to construct a representations from a given set of coordinates. We will also see that a positive representation defines a positive map from the Farey set of the surface to the flag variety, and that this notion can be generalized to closed surfaces

These notes constitute an introduction to some ideas of the definitions and properties of positive representations. We avoid technical difficulties in definitions and the proofs are only sketched here. For the correct definitions, and detailed proofs, one should obviously refer to the original articles of Fock and Goncharov [1, 2, 3].

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2. Framed representations

Let S be a compact orientable surface of genus g with $s \geq 1$ open discs removed. The standard presentation of the fundamental group is given by :

$$\pi_1(S) = \langle A_1, B_1, \dots, A_g, B_g, C_1, \dots, C_s \mid \prod_{i=1}^{g} [A_i, B_i] \prod_{j=1}^{s} C_j \rangle$$

where the C_j correspond to the homotopy type of a curve going aroung the j-th hole. Let G be a Lie group. The moduli space of representations into G is

$$\mathcal{R}_G(S) = \mathrm{Hom}(\pi_1(S), G)/G$$

where the action of G is by conjugation. (Note that one should take the GIT quotient instead of the topological quotient to make this space Hausdorff, but we will avoid this discussion in order to simplify this exposition).

We are going to describe a ramified cover of the space $\mathcal{R}_G(S)$ which will be the space of *framed representations* and will be denoted $\mathfrak{X}_G(S)$. The additional data is given by choices of flags in $\mathbb{C}^m$, and hence we first recall the necessary vocabulary on flags in a vector spaces.

2.1. **Flags.**

Definition 2.1. *A (complete) flag F in a finite dimensional vector space V is an increasing sequence of subspaces of V such that*

$$\{0\} = F_0 \subset F_1 \subset \cdots \subset F_m = V$$

with $\dim F_k = k$*.*

A basis $(e_1, \dots, e_m)$ is said to be adapted to the flag if the first k elements provide a basis of F_k. The stabilizer subgroup $\mathrm{Stab}(F) \subset G$ of a complete flag is identified with the set of invertible upper triangular matrices with respect to any basis adapted to the flag.

Let $F = (F_1, \dots, F_m)$ and $F' = (F'_1, \dots, F'_m)$ be two complete flags. We say that the pair of flags is in generic position if for any k we have $F_k \cap F'_{m-k} = \{0\}$. We say that a k-uple of flag is in generic position, if any pair of flag is in generic position.

Note that for a generic k-uple of flags $(A^{(1)}, \dots, A^{(k)})$ in $\mathbb{C}^m$ and dimensions of subspaces $i_1, \dots, i_k$, the dimension of a direct sum is given by

$$\dim(A^{(1)}_{i_1} + \cdots + A^{(k)}_{i_k}) = \min(m, i_1 + \dots i_k)$$

Hence if $i_1 + \dots i_k \leq m$ then the sum is direct.

The group $G = \mathrm{PSL}(m, \mathbb{R})$ or $\mathrm{PGL}(m, \mathbb{C})$ (depending on the field of the vector space V) acts naturally on flags by left multiplication. Obviously this action is transitive. The group G also acts on k-uples of flags. The space of G-orbits of k-uples of flags is called the space of configuration of k flags in $\mathbb{C}^m$, also denoted $\mathrm{Conf}_k(\mathbb{C}^m)$. If we restrict to k-uples of flags in generic position, the space is denoted $\mathrm{Conf}^*_k(\mathbb{C}^m)$.

Lemma 2.2. *The group G acts transitively on couple of flags in generic position. Moreover, the stabilizer of a couple of flag is the group of diagonal matrices in a basis adapted to one of the flag.*

Proof. Left as an exercise $\square$

2.2. **Definition.**

Definition 2.3. *A framed representation of S is given by the data of a representation $\rho \in \mathrm{Hom}(\pi_1(S), \mathrm{PGL}(m,\mathbb{C}))$ and flags $(F_1, \dots, F_s)$ in $\mathbb{C}^m$ associated to each connected component of ∂S, such that if $C_j \in \pi_1(S)$ corresponds to a boundary curve, then the corresponding flag is invariant by $\rho(C_j)$.*

The group $\mathrm{PGL}(m,\mathbb{C})$ acts naturally on framed representations by conjugation on the representation and left multiplication on the flags. Hence we define the moduli space of framed representation as the quotient of the set of framed representations by the action of G, and we denote it by $\mathfrak{X}_{\mathrm{PGL}(m,\mathbb{C})}(S)$.

A generic element $\rho(C_j) \in \mathrm{PGL}(m,\mathbb{C})$ corresponding to a boundary component have m invariant eigendirections. Hence there are $m!$ choices of invariant flags over a boundary circle corresponding to all possible ordering of the projective basis of eigendirections. However, when the eigenvalues are not all distinct (for example when the element is parabolic), then there are fewer choice of invariant flags, (and possibly only one). Hence, the space $\mathfrak{X}_{\mathrm{PGL}(m,\mathbb{C})}(S)$ is a ramified cover of $\mathbb{R}_{\mathrm{PGL}(m,\mathbb{C})}(S)$.

Remark 2.4. *One can extend this definition to surfaces with finite number of marked points removed on the boundary. In this case, the flags attached to the connected component of the boundary that are segments have no restrictions. This could allow us to identify a configuration of k flags as a framed representation of a disc with k marked points on the boundary.*

2.3. **Triangulations of surfaces.** Let S be a surface of genus g with s boundary components, such that $\chi(S) = 2-2g-s < 0$. Shrink all boundary components to points. Then the surface S can be cut into triangles with vertices at the shrunk boundary components. We call it an *ideal triangulation* of S.

Let T be a triangulation, and $V(T), E(T), F(T)$ the set of vertices, arcs, external arcs and faces of the triangulation T. Simple computation using the Euler characteristic give

(1) $|V(T)| = s$
(2) $|E(T)| = 6g - 6 + 3s$
(3) $|F(T)| = 4g - 4 + 2s$

3. Coordinates on moduli space of framed structure

We are now going to describe coordinates for the spaces $\mathfrak{X}_G(S)$. One of the main idea to find coordinates is to use a triangulation and decompose the space into a product of simpler space, that will correspond to moduli spaces of triples of flags and quadruples of flags. Hence, the coordinates are divided into two groups :

- The triangle invariants which corresponds to coordinates on $\mathrm{Conf}_3^*(\mathbb{C}^m)$.
- The edge invariants which parametrize the possible gluing of two triple of flags along an edge. We will show that this is parametrized by a maximal torus in $\mathrm{PGL}(m,\mathbb{C})$.

Theorem 3.1. *Let S be a surface and T an ideal triangulation. There is a birational isomorphism :*

$$\pi_T : \mathfrak{X}_G(S) \longrightarrow \prod_{t\in F(T)} (\mathrm{Conf}_3^*(\mathbb{C}^m)) \times \prod_{e\in E(T)} H$$

where H is a maximal torus in $\mathrm{PGL}(m,\mathbb{C})$.

The demonstration of this theorem will occupy this section where the map π_T will be explicitly constructed, and the next section where π_T^{-1} will be determined.

We will first study the classical case of $\mathrm{PGL}(2,\mathbb{C})$ where the invariants will be related to Thurston shearing coordinates on Teichmüller space. Then, we will see the case of $\mathrm{PSL}(3,\mathbb{C})$ where one has to understand the triple-ratio of a triple of flags in $\mathbb{C}^3$. The general coordinates will easily be constructed from this two basic cases.

3.1. **Cross-ratios in the $m = 2$ case.** A flag in $\mathbb{C}^2$ is simply given by a direction in $\mathbb{C}^2$ or equivalently a point in $\mathbb{C}P^1$. A classical result states that $\mathrm{PGL}(2,\mathbb{C})$ acts transitively on generic triples of flags in $\mathbb{C}P^1$, so there is only one orbit of generic triples of flags. Hence, the space $\mathrm{Conf}_3^*(\mathbb{C}^2)$ is reduced to a single point. Note that a maximal torus in $\mathrm{PGL}(2,\mathbb{C})$ is isomorphic to $\mathbb{C}^*$. Hence we simply have to find a unique coordinate for the configuration of four flags in $\mathbb{C}P^1$.

Let A, B, C, D be four flags in $\mathbb{C}^2$ in generic position, assimilated to points in $\mathbb{C}P^1$. We can consider that these four flags are attached to the vertices of a quadrilateral. Take the triangulation of this quadrilateral with an edge joining (the vertices associated to) A and C. We are going to associate a coordinate to this configuration and attach it to the edge AC.

The coordinate is given by he cross ratio of the 4 points (A, B, C, D) (in the correct order) in $\mathbb{C}P^1$. denoted by

$$[A, B, C, D] = \frac{(A - D)(B - C)}{(A - C)(B - D)}$$

This definition is equivalent to the following one : send the first triangle (A, B, C) to $0, -1, \infty$ (which is possible as the action is transitive) and then the fourth vertex is sent to a point $x \in \mathbb{C}P^1$ which is exactly this cross-ratio. Note that this definition is not exactly the standard definition, but there is a sign-change. This is due to the fact that we want the cross-ratio to be positive in specific situations.

Now let S be a general surface. Let $[\hat{\rho}]$ be a conjugacy class of framed representation and T a triangulation of S. We can lift this triangulation to a triangulation $\widetilde{T}$ of the universal cover $\widetilde{S}$. Starting with a choice of an arbitrary choice of a flag at a vertex of $\widetilde{T}$ we can associate to each vertex of the triangulation $\widetilde{T}$ a flag in a ρ-equivariant way. We can associate a coordinate for each edge of this triangulation by computing the cross-ratio of the four-points defined by the edge. We can restrict ourselves to a fundamental domain of the universal cover, as the ρ-equivariance ensures that the coordinates are invariant by deck transformation. Hence we have the map

$$\Phi_T : \mathfrak{X}_{\mathrm{PGL}(2,\mathbb{C})}(S) \longrightarrow (\mathbb{C}^*)^{|E(T)|}$$

that will give a set of coordinates on the moduli space of framed representations.

3.2. **Triple ratio in the $m = 3$ case.** We need to understand the moduli space of configuration of triples of flags in $\mathbb{C}^3$, and find coordinates on it. A flag in $\mathbb{C}^3$ is given by a line and a plane, or equivalently by a point and a line in $\mathbb{C}P^2$. Heuristically, the space of triples of flags is of dimension 9 (dimension 3 for each flag) while the group $G = \mathrm{PGL}(m,\mathbb{C})$ is of dimension 8. Hence the moduli space of configuration is of dimension 1, and we shall need only one invariant to describe it. This invariant is given by the triple ratio defined as follows.

Let $A = (A_1, A_2)$, $B = (B_1, B_2)$ and $C = (C_1, C_2)$ be a triple of flag, where $\dim A_i = i$. Choose v_A, v_B, v_C some direction vectors for the lines A_1, B_1 and C_1. and let $f_a, f_b, f_c \in (\mathbb{C}^3)^*$

be linear forms representatives defining the planes A_2, B_2 and C_2. The triple ratio is given by :

$$X := r_3(A,B,C) = \frac{f_a(v_b)f_b(v_c)f_c(v_a)}{f_a(v_c)f_b(v_a)f_c(v_b)}$$

This does not depend on the choices of the vectors or the linear form (easy exercise). This quantity is a complete invariant of generic configuration of three flags. Namely, if two flags A,B,C and A',B',C' have the same triple ratio, then there exists a unique element $g \in \mathrm{PGL}(m,\mathbb{C})$ such that $g(A) = A'$, $g(B) = B'$ and $g(C) = C'$ (left as an exercise).

Now we can look at the moduli space of configuration of four flags in $\mathbb{C}^3$. Let A,B,C,D be four flags in $\mathbb{C}^3$ in generic position and such that we have $r_3(A,B,C) = X$ and $r_3(A,C,D) = Y$. We want to construct invariants using cross-ratio of quadruples of points in $\mathbb{C}P^1$ associated to the edge AC. Note that four planes in $\mathbb{C}^3$ sharing a line in common define four lines in $\mathbb{C}^2$ by projection, and hence can be turned into four points in $\mathbb{C}P^1$.

The four planes $(A_2, A_1 \oplus B_1, A_1 \oplus C_1, A_1 \oplus D_1)$ all contain the line A. Hence, we can define their cross-ratio Z and get an element in $\mathbb{C}^*$. Similarly the four planes $(C_2, C_1 \oplus D_1, C_1 \oplus A_1, C_1 \oplus B_1)$ all contain the line C_1. So we can also define their cross-ratio W.

Proposition 3.2. *The map*

$$\begin{aligned} \mathrm{Conf}_4^*(\mathbb{C}^3) &\longrightarrow (\mathbb{C}^*)^4 \\ (A,B,C,D) &\longmapsto (X,Y,Z,W) \end{aligned}$$

is a birational isomorphism.

Now let S be a general surface. Let $[\hat{\rho}]$ be a conjugacy class of framed representation and T an ideal triangulation of S. Again on each vertex of the lifted triangulation, we can associate the flag of the corresponding element of ∂S in a ρ-equivariant way. Hence, for each triangle of the triangulation we can associate a coordinate by computing the triple-ratio of the triple of flags associated to the vertices. And for each edge of the triangulation, we can associate the two coordinates corresponding to the two cross-ratio as above. Hence we have a map

$$\Phi_T : \mathfrak{X}_{\mathrm{PGL}(3,\mathbb{C})}(\hat{S}) \longrightarrow (\mathbb{C}^*)^{2|E(T)|+|F(T)|}$$

which gives $16g - 16 + 8s$ coordinates for the moduli space of framed representations.

3.3. **Coordinates in the general case.** When $m \geq 3$ the same ideas are used to construct coordinates. The triangle invariants are computed by restricting the flags in $\mathbb{C}^m$ to $\mathbb{C}^3$ and the edge invariants are computed by finding four planes sharing a line in $\mathbb{C}^3$.

3.3.1. *Invariants of triples of flags.* Let $m \geq 3$ and A, B and C be a generic triple of flags in $\mathbb{C}^m$. Denote by A_k the subspace of dimension k in the flag A. The main idea to get invariants of triple of flags is to extract from the three flags in $\mathbb{C}^m$, a certain number of triples of flag in $\mathbb{C}^3$ in a clever way.

Let $i,j,k \geq 1$ such that $i+j+k = m$. The direct sum $W_{i,j,k} = A_{i-1} \oplus B_{j-1} \oplus C_{k-1}$ is a subspace of dimension $m-3$. Hence the quotient $V_{i,j,k} = \mathbb{C}^m/W$ is isomorphic to $\mathbb{C}^3$.

We can take the projection of the subspace A_i and A_{i+1} on V, and this give a flag $A^{(i,j,k)} = (\overline{A}_i, \overline{A}_{i+1})$ in $V_{i,j,k}$. Similarly, we get two other flags $B^{(i,j,k)}$ and $C^{(i,j,k)}$ by projecting B and C on $V_{i,j,k}$. Hence we have a triple of flags in $\mathbb{C}^3$ and hence we can compute :

$$X_{i,j,k}(A,B,C) = r_3(A^{(i,j,k)}, B^{(i,j,k)}, C^{(i,j,k)})$$

Obviously, this quantity is an invariant of triples of flags in $\mathbb{C}^m$. For each $i+j+k=m$ we get another invariant. This gives $\frac{(m-1)(m-2)}{2}$ coordinates in $\mathbb{C}^*$. We see that this number should be equal to the dimension of the space of configuration of three flags in $\mathbb{C}^m$. Indeed we have

Proposition 3.3. *The map*

$$\begin{aligned} \mathrm{Conf}_3(\mathbb{C}^m) &\longrightarrow (\mathbb{C}^*)^{\frac{(m-1)(m-2)}{2}} \\ (A,B,C) &\longmapsto \{X_{i,j,k}(A,B,C) | i,j,k \geq m, \ \ \text{and } i+j+k=m, \} \end{aligned}$$

is a birational isomorphism.

3.3.2. *Invariants of quadruples of flags.* Let A, B, C, D be a quadruple of flags in $\mathbb{C}^m$. The main idea is to extract several quadruples of lines in $\mathbb{C}^2$, that we will associate to this edge

Let $i, j \geq 1$ such that $i+j=m$. The direct sum $U_{i,j} = A_{i-1} \oplus C_{j-1}$ is a subspace of dimension $m-2$. Hence the quotient $V_{i,j} = \mathbb{C}^m / U_{i,j}$ is isomorphic to $\mathbb{C}^2$

We can take the projection of A_i, B_1, C_j and D_1 that we denote $A^{(i,j)}$, $B^{(i,j)}$, $C^{(i,j)}$ and $D^{(i,j)}$ on this subspace. And this will give a quadruple of lines in $\mathbb{C}^2$ so we can compute :

$$X_{i,j}(A,B,C,D) = [A^{(i,j)}, B^{(i,j)}, C^{(i,j)}, D^{(i,j)}]$$

This quantity is an invariant of quadruple of flags in $\mathbb{C}^m$. For each $i+j=m$ we get another invariant. This gives $m-1$ coordinates in $\mathbb{C}^*$ that we attach to an edge. This is exactly the dimension of a maximal torus H in $\mathrm{PGL}(m, \mathbb{C})$.

3.3.3. *Coordinates for a general surface.* Let S be a general surface, with a triangulation. To visualize easily all the invariants and their relation, we use an m-triangulation of the surface S, where each original triangle is divided into more triangles as seen on the figure below. Each vertex of this m-triangulation correspond to a triple (i,j,k) such that $i,j,k \geq 0$ are integers and $i+j+k=m$.

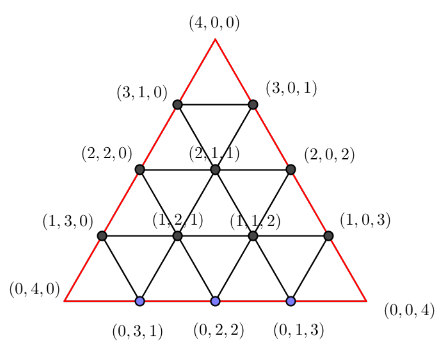

Figure 1. 4-triangulation : Each point corresponds to a triple (i,j,k) where at least two are non-zero

For each point, except the vertices of the original triangle, on this m-triangulation, we associate a coordinate. Namely, to each vertex in the interior of a triangle corresponds a triple $i,j,k \geq 1$ such that $i+j+k=m$ so we can associate a triangle invariant. and for each vertex in the interior of a side of the triangle corresponds a couple $i,j \geq 1$ such that $i+j=m$ so we can associate an edge invariant.

Let $N(S,m)$ the number of vertices of this m-triangulation of the surface. We get

$$N(S,m) = |F(T)|\left(\frac{(m-1)(m-2)}{2}\right) + |E(T)|(m-1)$$

So with the same principle as before, this gives a map

$$\Phi_T : \mathfrak{X}_{\mathrm{PGL}(m,\mathbb{C})}(S) \longrightarrow (\mathbb{C}^*)^{N(S,m)}$$

In the next section, we construct the inverse map Φ_T^{-1}. We start with coordinates in $(\mathbb{C}^*)^{N(S,m)}$ and we find an element of $\mathfrak{X}_{\mathrm{PGL}(m,\mathbb{C})}(S)$.

4. Construction of a framed representation from coordinates

4.1. **General Strategy.** Starting from the triangulation, we construct a graph Γ embedded into the surface by drawing small edges transversal to each side of the triangles and inside each triangle connect the ends of edges pairwise by three more edges. Orient the small edges in the triangles in counterclockwise direction and the long edges crossing the triangulation in an arbitrary way (see Figure 2).

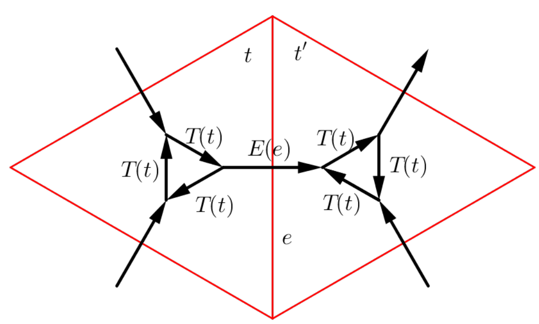

Figure 2. The oriented graph dual to the triangulation

To construct a framed representation, we assign to every oriented edge $e \in \Gamma$, an element in $\mathrm{PGL}(m,\mathbb{C})$ that will depend on the coordinates associated to it.

For any long edge $e \in \Gamma$, we associate a matrix $E(\{X_{i,j}\})$ which will depend on the edge coordinates $X_{i,j}$ with $i+j=m$ of the corresponding edge of the triangulation. And to any small edge $e \in \Gamma$, and we associate the matrix $T(\{X_{i,j,k}\})$ which will depend on triangle coordinates of the triangle in which it stands.

Given this, for any closed path $\gamma \in \pi_1(S)$ on the surface, there is a path on Γ homotopic to it. So one can assign to γ an element of $\mathrm{PGL}(m,\mathbb{C})$ by multiplying the group elements (or their inverse if the path goes along the edge against its orientation) assigned to the edges followed by the path. When the matrices T and E are well constructed, this defines a framed representation of the fundamental group of S, with the appropriate coordinates.

The main idea is the following. Let t be a triangle with flags (A, B, C) at each of its vertices and with invariants $X_{i,j,k}$. Then the matrix $T(t)$ associated to this triangle will correspond to the element of $\mathrm{PGL}(m, \mathbb{C})$ that will perform the cyclic permutation of the flags $(A, B, C) \mapsto (B, C, A)$.

Similarly, if (A, B, C, D) is a quadruple of flags with coordinates $X_{i,j}$ associated to the edge (A, C). Then the matrix $E(e)$ will correspond to the element of $\mathrm{PGL}(m, \mathbb{C})$ that sends (A, C) to (C, A) and moreover sends B_1 to D_1.

We will see explicit constructions in the cases $m = 2$ and $m = 3$. The explicit construction in the general case is a little more difficult to compute but follow the same principle.

4.2. **The case** $m = 2$**.** Let (A, B, C) be a triple of flags in $\mathbb{C}^2$ or equivalently, three points in $\mathbb{C}P^1$. For the matrix T, we take the element sending the triple (A, B, C) to (B, C, A), and we can choose (A, B, C) to be $(0, \infty, -1)$ so that

$$T = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$

Now let (A, B, C, D) be a quadruple of points such that the coordinates associated to the edge AC is equal to $z \in \mathbb{C}^*$. For convenience, we may take $(A, B, C, D) = (0, -1, \infty, z)$. Now the matrix $E(z)$ should be the element sending $(0, \infty)$ to $(\infty, 0)$ and moreover sending -1 to z. This is

$$E(z) = \begin{pmatrix} 0 & z \\ -1 & 0 \end{pmatrix}$$

This will gives the representation of $\pi_1(S)$.

To recover the framing over holes, notice that for any coordinate z a product of the form $E(z)T$ is a lower-triangular matrix (while $E(z)T^{-1}$ is upper lower triangular. When following a path around the boundary we get only one type of product, because we always turn right (resp. left), and hence only get T (resp. T^{-1}) in the formulas. So the monodromy $\rho(C_j)$ is upper-triangular (or lower-triangular), which gives a preferred eigendirection for each boundary, and hence an invariant flag.

4.3. **The case m=3.** Let A, B, C be a triple of flag in $\mathbb{C}^3$, with triple ratio $r_3(A, B, C) = X$. We want to construct an element g that will send (A, B, C) on (B, C, A).

By the transitivity of the G-action on couple of flags, we can assume without loss of generality that we have a basis (e_1, e_2, e_3) such that :

$$A_1 = e_1, \qquad A_2 = e_1 \oplus e_2, \qquad C_1 = e_3, \qquad C_2 = e_3 \oplus e_2$$

and moreover such that

$$B_1 = e_1 - e_2 + e_3$$

In this case, a linear form generating B_2 is given by $Xe_1^* + (1 + X)e_2^* + e_3^*$. So the desired matrix in the basis (e_1, e_2, e_3) is given by :

$$T(X) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ X & 1 + X & 1 \end{pmatrix}$$

Now let A, B, C, D be four flags in $\mathbb{C}^3$ such that the invariant associated to the edge AC are Z and W. Again, we can assume that we have a basis such that A and C are the standard flags $(e_1, e_1 \oplus e_2)$ and $(e_3, e_3 \oplus e_2)$ respectively. And with B_1 generated by $e_1 - e_2 + e_3$. Then by definition of the two cross-ratios Z and W we have that the vector $Z^{-1}e_1 + e_2 + We_3$ is a direction vector of D_1. Then in this basis, the element sending (A, C) to (C, A) and B_1 to D_1 is given by

$$E(Z, W) = \begin{pmatrix} 0 & 0 & Z^{-1} \\ 0 & -1 & 0 \\ W & 0 & 0 \end{pmatrix}$$

Remark 4.1. *The general case is obtained by the same principle. However the matrices and their computations are a little too long to be included in these notes. The interested reader can find all the details in* [1] **9.8**.

5. Positive Representations

We now have all the tools to define the set of positive representations as a subset of the space of framed representations.

5.1. **Positive part of the moduli space.** The coordinates constructed depend on the chosen triangulation. An important thing is to understand how these coordinates change when we choose a different triangulation of the surface S. A classical result states that two triangulation are related by a sequence of flips. Hence we need to understand what happens to the coordinates when we perform a flip along an edge of a triangulation.

In the case $m = 2$. It is an easy exercise to recover the following formulas in Figure 3.

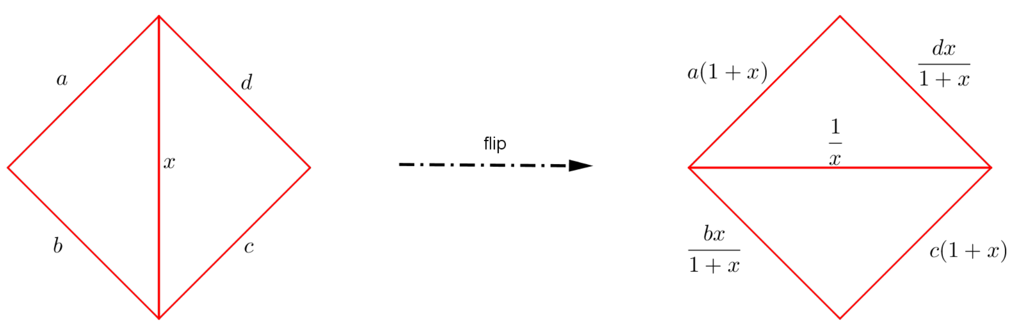

Figure 3. Change of coordinates after a flip

5.1.1. *Cluster mutations.* There is a simple interpretation of these coordinate changes in terms of cluster algebra. From a triangulation of the surface S, we can define an oriented graph (a quiver) associated to it (see Fig. 4). Flipping along an arc of the triangulation is equivalent to a performing a mutation of the quiver along a vertex. Such mutation are defined for any quiver as follows :

Let Q be a quiver and k one of its vertex, the mutated quiver $\mu_k(Q)$ is obtained by performing the three following steps :

(1) For each path of length two of the form $i \longrightarrow k \longrightarrow j$, add an arrow $i \longrightarrow j$
(2) Reverse each arrow incident to the vertex k.

(3) Erase any 2-cycles that could have been created by previous steps.

The new coordinates are given by the so-called mutations formulas, which were discovered by Fomin and Zelevinsky in a much broader context, and depend only on the quiver. If $(x_1, \ldots, x_n)$ are the coordinates associated to each vertex of the quiver Q. Let $\varepsilon(i,j)$ be the number of oriented arrows between the vertex i and j. Then the new coordinates $(x'_1, \ldots, x'_n)$ of the quiver $\mu_k(Q)$ are given by

$$x'_j = \begin{cases} \dfrac{1}{x_j} & \text{if } j = k, \\ x_j(1+x_k)^{\varepsilon(j,k)} & \text{if } \varepsilon(j,k) \geq 0, \\ x_j(1+x_k^{-1})^{\varepsilon(j,k)} & \text{if } \varepsilon(j,k) < 0 \end{cases}$$

For a detailed exposition of the relation between cluster algebras and triangulations of hyperbolic surfaces, one should refer to the article [4] of Fomin, Shapiro and Thurston.

5.1.2. *Flips as sequence of mutations.* In the general case, the explicit formulas for the change of coordinates after a flip could be computed directly but would be quite heavy. However, it is again possible to interpret these changes in terms of cluster mutation. We can associate a quiver to any m-triangulation of the surface as indicated in the figure 4. In this case, changing the triangulation is equivalent to a particular sequence of quiver mutations, and the change of coordinates corresponds to the associated cluster mutations (see Chapter **10** of [1]).

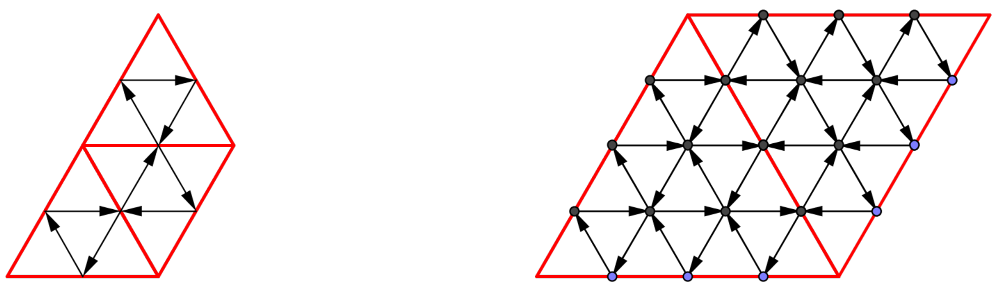

Figure 4. Quiver associated to a triangulation and to an m-triangulation of the surface

We notice that the formulas for cluster mutations are subtraction-free. Hence, any change of triangulation gives a subtraction-free change of coordinates. This defines a *positive atlas* on the space of framed structure (the actual definition of a positive atlas is not needed). Hence, it is clear that if a framed representation has positive coordinates in a given triangulation, then it will have positive coordinates with respect to any triangulation. This allows us to define

Definition 5.1. *A framed representation is said to be* positive *if its coordinates with respect to one (and hence any) triangulation are all real positive.*

The set of positive framed representation is denoted $\mathfrak{X}_G^+(S)$.

A positive representation is just the image of a positive framed representation through the forgetful map $\mathfrak{X}_G(S) \to \mathcal{R}_G(S)$.

In the next two paragraphs, we will give sketches of proofs of the fundamental properties of positive representations. Namely that they are faithful and discrete.

5.2. **Totally positive matrices and faithfulness of positive representations.**

Proposition 5.2. *Let $[\rho]$ be a positive framed representation. Then the corresponding representation ρ is faithful. Moreover, for any element $\gamma \in \pi_1(S)$ which is non-trivial and non-boundary, the element $\rho(\gamma)$ is positive hyperbolic.*

To prove this proposition, we must look at the construction of the framed structure from the coordinates. Let γ be a non-trivial non-boundary loop. Then the element $\rho(\gamma)$ is of the form

$$\rho(\gamma) = E(e_1)(T(t_1))^{\epsilon_1}E(e_2)(T(t_2))^{\epsilon_2}\cdots E(e_p)(T(t_p))^{\epsilon_p}$$

Suppose all the coordinates are positive. Then we notice that any matrix of the form $E(e)T(t)$ (resp. $E(e)(T(t))^{-1}$) will be a totally positive upper-triangular matrix (resp. totally positive lower triangular matrix). The product of totally positive matrices is a totally positive matrix. And totally positive matrices have the property to have real, positive and distinct eigenvalues. Hence, $\rho(\gamma)$ is positive hyperbolic. The faithfulness is easily deduced from that.

5.3. **Farey set and discreteness.** To prove discreteness, we need to introduce a new object.

Definition 5.3. *The Farey set of the surface S, denoted $\mathcal{F}_\infty(S)$ is a cyclic $\pi_1(S)$-set defined as follows.*

Shrink the holes of S to punctures and lift to the universal cover, which is an open disc. The set $\mathcal{F}_\infty(S)$ is the set of lifts of punctures which is a countable subset on the boundary of the disc. The group $\pi_1(S)$ acts naturally by deck transformation, and the cyclic structure is given by the cyclic structure on the boundary of the disc.

We can defined framed representations using only maps from the Farey set to the flag variety as follows as there is a natural bijection between $\mathfrak{X}_{\mathrm{PGL}(m,\mathbb{C})}(S)$ and the set of ρ-equivariant maps

$$\beta_\rho : \mathcal{F}_\infty(S) \longrightarrow \mathrm{Flag}(\mathbb{C}^m)$$

modulo conjugation by $\mathrm{PGL}(m, \mathbb{C})$.

Definition 5.4. *A triplet of flags (A, B, C) is* positive *if it is equivalent (modulo the action of $\mathrm{PGL}(m, \mathbb{C})$) to a triple $(F^+, F^-, u \cdot F^-)$ where F^+ and F^- are the standard flag and u is a totally positive upper triangular matrix.*

A configuration of flag $(F_1, \dots, F_n)$ is positive if any oriented triplet of flag is positive.

If $[\rho]$ is a positive framed representation, then the map β_ρ takes values in $\mathrm{Flag}(\mathbb{R}^m)$ and is a positive map in the following sense : for any finite cyclic subset $(x_1, \dots, x_k) \in \mathcal{F}_\infty(S)$, the configuration of flags $(\beta(x_1), \dots, \beta(x_n))$ is positive.

Proposition 5.5. *We have an identification of $\mathfrak{X}^+_{\mathrm{PGL}(m,\mathbb{C})}(S)$ and the set of (ρ, β) where ρ is a representation and $\beta : \mathcal{F}_\infty(S) \to \mathrm{Flag}(\mathbb{R}^m)$ is a positive $(\pi_1(s), \rho)$-equivariant map, modulo conjugation.*

This characterization of positive representation can be extended to surfaces without boundaries. In this case, there are no positive structure or coordinates but the properties of positive representations remain.

The identification also allows us to prove the following theorem

Theorem 5.6. *Positive representations are discrete.*

Proof. Let ρ be a positive representation. Let $(a, b, c) \in \mathcal{F}_\infty(S)$ be an ideal triangle of the triangulation of S. The flag $\beta(b)$ belongs to the set

$$\mathcal{D}_- = \{F \in \mathrm{Flag}(\mathbb{R}^m) | (\beta(a), F, \beta(c)) \text{ is a positive configuration of flags}$$

This set is open and disjoint from

$$\mathcal{D}_+ = \{F \in \mathrm{Flag}(\mathbb{R}^m) | (\beta(a), \beta(c), F) \text{ is a positive configuration of flags}$$

Hence, there exist an open neighborhood $O \subset \mathrm{PSL}(m, \mathbb{R})$ such that for all $g \in O$, we have $g \cdot \beta(b) \notin \mathcal{D}_+$.

Now let $\gamma \in \pi_1(S)$ and suppose that the image $b' = \gamma \cdot b$ is contained in the segment $(a, c) \in \mathcal{S}^1$. As β_ρ is a positive map, the quadruple of flags $(\beta(a), \beta(b), \beta(c), \rho(\gamma)\beta(b))$ is positive. We deduce easily that $\rho(\gamma) \notin O$. $\square$

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