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Abstract

These are some brief notes on surface groups actions on complex hyperbolic space, mainly work of Domingo Toledo, and generalizations.

1 Hyperbolic space

We begin with motivation from hyperbolic space. First, the definition most useful to our generalization. Let q be the quadratic form on $\mathbb{R}^{n+1}$ given by

$$q(\alpha_1, \dots, \alpha_{n+1}) = \alpha_1^2 + \cdots + \alpha_n^2 - \alpha_{n+1}^2.$$

The associated symmetric matrix is

$$B = \mathrm{diag}(1, \dots, 1, -1),$$

and we also use B to denote the associated symmetric bilinear form on $\mathbb{R}^{n+1}$. The *Klein model* of hyperbolic space $\mathbf{H}^n$ is then

$$\mathcal{P}^n = \left\{ \text{lines } \ell \subset \mathbb{R}^{n+1} \ : \ q|_\ell \text{ is negative} \right\} \subset \mathbb{RP}^n.$$

The metric[1] is

$$d(x, y) = 2\cosh^{-1}\left(\frac{B(u, v)}{\sqrt{q(u)q(v)}} \right),$$

where $u, v \in \mathbb{R}^{n+1}$ are vectors representing the lines $x, y \in \mathbb{RP}^n$. Notice that $q(u), q(v) < 0$, but the square root is real.

[1] The factor of 2 makes the curvature -1.

Exercise 1. Show that d is well-defined.

Exercise 2. Identify $\mathcal{P}^n$ with the unit ball and upper half space models.

Exercise 3. Show that the ideal boundary of $\mathcal{P}^n$ is the set of *q-isotropic* lines, $\{\ell \subset \mathbb{R}^{n+1} \ : \ q|_\ell = 0\}$.

Exercise 4. Show that the (orientation-preserving) isometry group of $\mathcal{P}^n$ is exactly $\mathrm{PO}_0(n,1)$, the connected component of 1 in the group of projective transformations of $\mathbb{R}^{n+1}$ that fix q.

Exercise 5. When is $\mathrm{PO}(n,1)$ connected?

Now, we consider a (closed) surface group Σ (of genus $g \geq 2$) and homomorphisms $\Sigma \to \mathrm{PO}_0(n,1)$. We start with the case $n = 2$. Fix a hyperbolic structure, i.e., a discrete and faithful representation $\rho_0 : \Sigma \to \mathrm{PO}_0(2,1)$, and let $S = \mathbf{H}^2 / \rho_0(\Sigma)$. The following exercise (see [6]) is important motivations for our generalizations to complex hyperbolic space.

Exercise 6. Given $\rho : \Sigma \to \mathrm{PO}_0(2,1)$, build a ρ-equivariant $\mathbf{H}^2$-bundle E_ρ over S with fiber $\mathbf{H}^2$ with the following property. Let vol_ρ be the $\mathrm{PO}_0(2,1)$-invariant extension of the volume form on $\mathbf{H}^2$ to E_ρ by pushing it along fibers. Then given any section σ of E_ρ the number

$$\mathrm{vol}(\rho) = \int_S \sigma^* \mathrm{vol}_\rho$$

is independent of σ.

In this thesis [5], Goldman showed that $\mathrm{vol}(\rho)$ is related to the Euler number of a certain bundle. Using this, he proved the following.

Theorem 1. *Let Σ be the fundamental group of a closed surface S of genus $g \geq 2$, and $\rho : \Sigma \to \mathrm{PO}_0(2,1)$ any representation. Then*

1. $\mathrm{vol}(\rho) \in 2\pi\mathbb{Z}$;
2. $|\mathrm{vol}(\rho)| \leq 2\pi(g-1)$;
3. $|\mathrm{vol}(\rho)| = 2\pi(g-1) = \mathrm{vol}(S)$ *if and only if ρ is Fuchsian, i.e., ρ is the holonomy of a complete hyperbolic structure on S;*
4. $\mathrm{vol}(\rho_1) = \mathrm{vol}(\rho_2)$ *if and only if ρ_1 can be deformed into ρ_2, that is, if and only if ρ_1 and ρ_2 lie in the same connected component of* $\mathrm{Hom}(\Sigma, \mathrm{PO}_0(2,1))$.

This is a remarkable rigidity theorem, especially for something as non-rigid as the fundamental group of a surface. The representations with $|\mathrm{vol}(\rho)| < 2\pi(g-1)$ are very poorly understood. That the extremal representations are the most rigid and geometrically significant[2] is a recurring theme in these notes.

Before complexifying, we consider $\mathbf{H}^n$ for higher n. To start, there are some easy representations $\Sigma \to \mathrm{PO}_0(n,1)$, namely, representations that factor

$$\Sigma \to \mathrm{SO}_0(2,1) \to \mathrm{PO}_0(n,1)$$

given by a totally geodesic embedding of $\mathbf{H}^2$ into $\mathbf{H}^n$.

Exercise 7. Explain why $\mathbf{H}^2 \to \mathbf{H}^n$ gives an embedding $\mathrm{SO}(2,1) \to \mathrm{PO}(n,1)$ instead of $\mathrm{PO}(2,1) \to \mathrm{PO}(n,1)$. Hint: Think about how the scalar matrices in $\mathrm{O}(n,1)$ intersect, say, $\mathrm{SO}(2,1)$. Explain this in geometric terms via the normal bundle to $\mathbf{H}^2$ in $\mathbf{H}^n$.

Exercise 8. When does a representation $\rho : \Sigma \to \mathrm{PO}_0(2,1)$ lift to a representation $\Sigma \to \mathrm{SO}(2,1)$?

If the map $\Sigma \to \mathrm{SO}_0(2,1)$ is (the lift of) a Fuchsian representation of Σ, then we also call the representation $\Sigma \to \mathrm{PO}_0(n,1)$ Fuchsian. That is, Fuchsian representations of Σ into $\mathrm{PO}_0(n,1)$ are extensions of Fuchsian representations into $\mathrm{SO}_0(2,1)$ via a totally geodesic embedding $\mathbf{H}^2 \to \mathbf{H}^3$.

Exercise 9. Describe the totally geodesic embeddings of $\mathbf{H}^m$ into $\mathbf{H}^n$ ($m < n$) via subspaces of $\mathbb{R}^{n+1}$ on which the restriction of q is positive definite.

Fuchsian representations of Σ into $\mathrm{PO}_0(n,1)$ are, however, not rigid for $n \geq 3$. This is due to the existence of *bending deformations.* Here is the sketch:

1. Write Σ as an amalgamated product $G_1 *_{\langle\gamma\rangle} G_2$ over $\mathbb{Z}$ by splitting the surface S along an essential separating curve γ.
2. Take a Fuchsian representation $\rho : \Sigma \to \mathrm{SO}_0(2,1) \to \mathrm{PO}_0(n,1)$ with associated totally geodesic embedding $f : \mathbf{H}^2 \to \mathbf{H}^n$. Let $\widetilde{\gamma} \subset \mathbf{H}^2$ be the axis of γ.
3. Let $\mathcal{F}$ be a fundamental domain for the action of Σ on $\mathbf{H}^2$ such that $\widetilde{\gamma}$ cuts $\mathcal{F}$ into two subdomains $\mathcal{F}_1$ and $\mathcal{F}_2$ associated with each side of γ in S.
4. Let $f_t : \mathbf{H}^2 \to \mathbf{H}^n$ be a continuous family of totally geodesic embeddings ($t \in \mathbb{R}$) with $f_0 = f$ and $f_t(\widetilde{\gamma}) = f_0(\widetilde{\gamma})$.

[2] I think it's fair to say that the Fuchsian representations into $\mathrm{PO}_0(2,1)$ are the most significant representations of Σ.

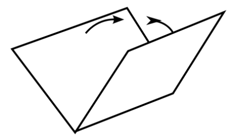

Figure 1: Local picture of a bending with $\widetilde{\gamma}$ at the crease, $\mathcal{F}_1$ to the left, and $\mathcal{F}_2$ to the right. It is from the wikipedia.com page on origami folds. (Ignore the arrows.)

5. Use $f(\mathcal{F}_1)$ and $f_t(\mathcal{F}_2)$ to define a one-parameter family ρ_t of representations $\Sigma \to \mathrm{PO}_0(n,1)$. By letting G_1 act with fundamental domain[3] $\mathcal{F}_1$ and G_2 with fundamental domain $\mathcal{F}_2$. This defines representations $G_j \to \mathrm{SO}_0(2,1) \to \mathrm{PO}_0(n,1)$.
6. By the universal property of amalgamated products, the representations $G_j \to \mathrm{PO}_0(n,1)$, which agree on γ, extend to a unique homomorphism $\rho_t : \Sigma \to \mathrm{PO}_0(n,1)$.

Exercise 10. Prove that, for t sufficiently small, ρ_t is discrete, faithful and, most importantly, *not Fuchsian.* Hint: Show that the Zariski closure of $\rho_t(\Sigma)$ is bigger than $\mathrm{SO}_0(2,1)$.

The important thing to take from this is that Fuchsian representations are highly non-rigid. In other words, one can easily deform Fuchsian representations into non-Fuchsian representations.

Exercise 11. Describe these deformations in terms of conjugation of $\rho(G_2)$ by the centralizer of $\rho(\gamma)$ inside $\mathrm{PO}_0(n,1)$.

2 Complex hyperbolic space

The following is the complexification of the Klein model described above. Also see [7]. Let h be the hermitian form on $\mathbb{C}^{n+1}$ given by

$$h(\alpha_1,\dots,\alpha_{n+1}) = |\alpha_1|^2 + \cdots + |\alpha_n|^2 - |\alpha_{n+1}|^2.$$

The associated hermitian matrix is

$$B = \mathrm{diag}(1,\dots,1,-1),$$

and we also use B to denote the associated sesquilinear form on $\mathbb{C}^{n+1}$. The *Klein model* of complex hyperbolic space $\mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$ is then

$$\mathcal{CP}^n = \big\{\text{complex lines } \ell \subset \mathbb{C}^{n+1} \ : \ h|_\ell \text{ is negative}\big\} \subset \mathbb{CP}^n.$$

[3]These are actually convex cores for fundamental domains.

The metric is

$$d(x,y) = 2\cosh^{-1}\left(\frac{B(u,v)}{\sqrt{h(u)h(v)}}\right),$$

where $u, v \in \mathbb{C}^{n+1}$ are vectors representing the lines $x, y \in \mathbb{CP}^n$.

Exercise 12. Take real parts and show that there is a totally geodesic embedding $\mathbf{H}^m \to \mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$ for all $m < n$. Hint: Prove that your map $\mathbf{H}^m \to \mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$ is totally geodesic by showing that it is the fixed point set of an involution.

Exercise 13. Show that $\mathcal{CP}^n$ is holomorphically equivalent to the unit ball in $\mathbb{C}^n$. Conclude that $\mathbf{H}^1_{\mathbb{C}}$ is the Poincaré disk model of $\mathbf{H}^2$.

Exercise 14. Show that the ideal boundary of $\mathcal{CP}^n$ is the set of h-*isotropic* lines, $\{\ell \subset \mathbb{C}^{n+1} \ : \ h|_\ell = 0\}$.

Exercise 15. Show that the (holomorphic) isometry group of $\mathcal{CP}^n$ is exactly $\mathrm{PU}(n,1)$, the group of projective transformations of $\mathbb{C}^{n+1}$ that fix h. What is the full isometry group?

Exercise 16. For $n = 2$, show that the ideal boundary is the one-point compactification of the 3-dimensional homogeneous space Nil. (Hint: Show that the stabilizer of a point on the boundary has a natural identification with Nil.)

Exercise 17. Show that the totally geodesic subspaces of $\mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$ are either $\mathbf{H}^m$ $(m \leq n)$ or $\mathbf{H}^m_{\mathbb{C}}$ $(m < n)$.

The most important thing to take away from the above is that there are two types of totally geodesic embedding $\mathbf{H}^2 \to \mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$. First are the $\mathbb{R}$-Fuchsian embeddings, which come from 3-dimensional real subspaces of $\mathbb{C}^{n+1}$ on which h defines a quadratic form of signature $(2,1)$. Second are the $\mathbb{C}$-Fuchsian representations, which come from 2-dimensional complex subspaces of $\mathbb{C}^{n+1}$ on which h has signature $(1,1)$; these are associated with holomorphic totally geodesic embeddings of the Poincaré disk into $\mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$.

Exercise 18. Show that $\mathbb{R}$-Fuchsian representations of a closed surface group Σ into $\mathrm{Isom}(\mathbf{H}^{\mathrm{n}}_{\mathbb{C}})$ are not rigid for $n \geq 3$. Hint: Use $\mathbf{H}^n$.

Exercise 19. Show that two copies of $\mathbf{H}^1_{\mathbb{C}}$ inside $\mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$ are either disjoint or intersect in a single point. Use this to explain why there is not version of the bending trick from §1 for $\mathbb{C}$-Fuchsian representations. Hint: The intersection is (1) totally geodesic and (2) a complex analytic subspace.

One can also show that $\mathbb{R}$-Fuchsian representations are not rigid when $n = 2$, but that takes some more work [9]. We will see in §4 that $\mathbb{C}$-Fuchsian representations are indeed rigid. In fact, they satisfy extremal properties for the *Toledo invariant*, and are analogous to the Fuchsian representations in Theorem 1.

3 The Gromov Norm

Before we define the Toledo invariant, we need the Gromov norm. This will give us the bounded cohomology class that leads to our replacement for $\mathrm{vol}(\rho)$ in §1. You should know something about this anyways. See [8] for tons more. We begin with the treatment of [3].

Let X be a topological space. For a singular chain $z = \sum a_i \sigma_i \in C_*(X)$, the L^1 norm is given by $\|z\|_1 = \sum |a_i|$. We then have a pseudo-norm on $H_*(X, \mathbb{R})$ (here, a map $H_*(X, \mathbb{R}) \to \mathbb{R}^+ \cup \{0\}$ satisfying the usual properties) by

$$\|x\|_1 = \inf\{\|z\|_1 \ : \ z \text{ a chain representing } x\}.$$

Similarly, if $c \in C^*(X)$ is a singular cochain, we have the sup-norm

$$\|c\|_\infty = \sup\{|c(\sigma)| \ : \ \sigma \text{ a (singular) simplex on } T\}.$$

Then, for a class $\alpha \in H^*(X, \mathbb{R})$, we obtain a pseudo-norm (here, a map $H^*(X, \mathbb{R}) \to \mathbb{R} \cup \{\infty\}$ satisfying the usual properties) by

$$\|\alpha\|_\infty = \inf\{\|c\|_\infty \ : \ c \text{ a cochain representing } \alpha\}.$$

We leave some basic but fundamental properties as exercises.

Exercise 20. For all $x \in H_*(X, \mathbb{R})$ and $\alpha \in H^*(X, \mathbb{R})$,

$$|\alpha(x)| \leq \|x\|_1 \|\alpha\|_\infty.$$

Exercise 21. If $f : X \to Y$ is continuous, then $\|f^*\alpha\|_\infty \leq \|\alpha\|_\infty$ for all $\alpha \in H^*(Y, \mathbb{R})$ and $\|f_* x\|_1 \leq \|x\|_1$ for all $x \in H_*(X, \mathbb{R})$.

Exercise 22. If X is a metric space of nonpositive curvature, we need only consider *geodesic* simplices (i.e., simplices with geodesic edges).

Of fundamental importance for us is the following fact, proved in [8].

Proposition 2. *Let $[S]$ be the fundamental class of a closed Riemann surface S of genus $g \geq 2$. Then $\|[S]\|_1 = 4(g-1) = -2\chi(S)$.*

Proof. Choose a hyperbolic metric on S and represent $[S]$ as $\sum r_i \sigma_i$, where σ_i is a geodesic triangle on S. By Gauss–Bonnet,

$$\sum r_i \operatorname{vol}(\sigma_i) = -2\pi\chi(S).$$

Since all triangles have area at most π, we get

$$2\pi|\chi(S)| \leq \sum |r_i|\pi,$$

so $\|[S]\|_1 \geq 2|\chi(S)|$.

It remains to show that $\|[S]\|_1 \leq 2|\chi(S)|$. Choose the hyperbolic metric associated with the hyperbolic $2g$-gon. We can then triangulate S by $4g$ geodesic triangles, so $\|[S]\|_1 \leq 2|\chi(S)| + 2$. However, we can apply this same triangulation trick to a k-fold covering, which represents $k[S]$, i.e., the pullback to S of the fundamental class of the covering, by $2k|\chi(S)| + 2$ triangles. Thus

$$\|k[S]\|_1 \leq 2k|\chi(S)| + 2$$

which implies that

$$\|[S]\|_1 \leq 2|\chi(S)| + \frac{2}{k}.$$

As $k \to \infty$, we get $\|[S]\|_1 \leq 2|\chi(S)|$, and this proves the proposition. $\square$

4 The Toledo Invariant

For hyperbolic Riemann surfaces, there is a $PO_0(2,1)$-invariant 2-form ω on $\mathbf{H}^2$ that is a *Kähler form.* For any Riemann surface $S = \Gamma \backslash \mathbf{H}^2$, the volume form on S is (up to a constant) the projection $\omega_S \in H^2(S, \mathbb{R})$ of ω to S. Similarly, the metric on $\mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$ is associated with a $\mathrm{PU}(n,1)$-invariant 2-form ω on $\mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$. Given any closed complex hyperbolic n-manifold $M = \Gamma \backslash \mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$, the projection $\omega_M \in H^2(M, \mathbb{R})$ is a nontrivial 2-form such that $\omega_M^n \in H^{2n}(M, \mathbb{R})$ is a multiple of the volume form; in particular, it is nontrivial.

Let $\rho : \Sigma \to \mathrm{PU}(n,1)$ be a representation. We can then define a ρ-equivariant continuous map $f : \mathbf{H}^2 \to \mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$ using the ρ-orbit of a point along with negative curvature. We then define the Toledo invariant

$$\tau(\rho) = \int_S f^*\omega.$$

Exercise 23. Show that $\tau(\rho)$ is independent of the choice of f. Hint: Think about the associated bundle and Exercise 6.

We then have the following fundamental result, due to Toledo.

Theorem 3 (Toledo [10])**.** *Let Σ be a closed surface group of genus $g \geq 2$ and $\rho : \Sigma \to \mathrm{PU}(n,1)$ be a representation. Then*

$$|\tau(\rho)| \leq 2|\chi(S)|.$$

Sketch of proof. We give the proof from [3]. The original proof uses harmonic maps and the so-called Bochner method.

The first step is to prove that $\left|\int_\Delta \omega\right| \leq \pi$ for any triangle Δ in $\mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$. One difficulty worthy of mention is that $\mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$ has variable curvature, so there is a nontrivial moduli space of $\mathrm{PU}(n,1)$-isomorphism classes of triangles. The result is relatively clear when Δ is a geodesic triangle in a holomorphically embedded $\mathbf{H}^2 \subset \mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$. The key is proving that this is the extremal case[4].

It follows that $\|f^*\omega\|_\infty \leq \pi$. From the above properties of the Gromov norm,

$$|\tau(\rho)| = |f^*(\omega)([S])| \leq \|f^*\omega\|_\infty \|S\|_1 \leq 2|\chi(S)|,$$

which proves the theorem. □

Theorem 4 (Toledo [11])**.** *Let Σ be a closed surface group of genus $g \geq 2$ and $\rho : \Sigma \to \mathrm{PU}(n,1)$ a representation. Then $|\tau(\rho)| = 2|\chi(S)|$ if and only if ρ is Fuchsian with respect to a holomorphic totally geodesic embedding $\mathbf{H}^2 = \mathbf{H}^1_{\mathbb{C}} \to \mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$.*

Sketch of proof. The key point is proving that $\rho(\Sigma)$ fixes such a totally geodesic subspace. That ρ is Fuchsian basically follows from Theorem 1 along with properties of the Gromov norm. Using some technical estimates along with the strict negative curvature of $\mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$, one obtains a measurable mapping $d_\rho : \partial\,\mathbf{H}^2 \to \partial\,\mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$. One then proves that if Δ is a (possibly ideal) triangle in $\mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$ with $\int_\Delta \omega = \pi$, then Δ is an ideal triangle in $\mathbf{H}^1_{\mathbb{C}} \subset \mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$. Applying this to arbitrary triples of points on $d_\rho(\partial\,\mathbf{H}^2)$ shows that almost every triple of points lie on the ideal boundary of a fixed $\mathbf{H}^1_{\mathbb{C}} \subset \mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$. This line is $\rho(\Sigma)$-invariant, which gives the theorem. □

[4]For example, if Δ is a triangle in a 'real' totally geodesic $\mathbf{H}^2 \to \mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$, then $\omega|_\Delta \equiv 0$, so the integral is trivial. In a certain sense, all other cases interpolate between these two extremes.

Remark. The basic idea of the proof is related to certain proofs of many rigidity theorems, e.g., Mostow rigidity, in that the important technical step is producing a measurable mapping of ideal boundaries that satisfies nice properties.

5 Generalizations

Over the last ten or fifteen years, there has been a great deal of progress on various generalizations of Toledo's result. First, instead of a Riemann surface, i.e., a complex hyperbolic 1-manifold, we can try to generalize Toledo's results to higher-dimensional complex hyperbolic manifolds. One such result from the same era as Toledo's work was the following rigidity theorem of Goldman–Millson [4].

Theorem 5. *For any* $1 < m \leq n$*, let* $\Gamma < \mathrm{SU}(n-1,1)$ *be a torsion free lattice and* $\rho : \Gamma \to \mathrm{SU}(n-1,1) \to \mathrm{PU}(n,1)$ *be a representation given by factoring the natural representation of* Γ *into* $\mathrm{SU}(n-1,1)$ *into* $\mathrm{PU}(n,1)$ *via a totally geodesic embedding* $\mathbf{H}^{n-1}_{\mathbb{C}} \to \mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$*. Then* Γ *is infinitesimally rigid, i.e., any small deformation of* Γ *also stabilizes a totally geodesic* $\mathbf{H}^{n-1}_{\mathbb{C}} \subset \mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$*.*

In particular, the 'Fuchsian' representations of Γ form a connected component of $\mathrm{Hom}(\Gamma, \mathrm{PU}(n,1))$. Soon thereafter, Corlette proved the following generalization [2].

Theorem 6. *Let* M *be a compact complex hyperbolic manifold of (complex) dimension* m *and fundamental group* Γ*. If* $n > m$ *and* $\rho : \Gamma \to \mathrm{SU}(n,1)$ *is a representation with* $\mathrm{vol}(\rho) = \mathrm{vol}(M)$*, then* $\rho(\Gamma)$ *stabilizes a totally geodesic* $\mathbf{H}^{m}_{\mathbb{C}} \subset \mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$*.*

Here, $\mathrm{vol}(\rho)$ is the pullback of ω^m to $H^{2m}(M, \mathbb{R})$, where ω is the Kähler class on $\mathbf{H}^{\mathrm{n}}_{\mathbb{C}}$, so $\mathrm{vol}(\rho)$ generalizes the Toledo invariant. Since $\mathrm{vol}(\rho)$ varies among a discrete set (e.g., by relation with a characteristic class), this is a strong generalization of the Goldman–Millson result stated above. In particular, the infinitesimal rigidity of Goldman–Millson, analogously to the case of surface group representations, passes to the entire connected component of the 'Fuchsian' representations.

Returning to surface groups, there are two types of generalizations. First, one can allow noncompact surfaces, where similar rigidity results follow once one insists that certain regularity of the $\mathrm{PU}(n,1)$

representation (these are so-called *maximal* representations). Instead, one can replace $\mathbf{H}^n_{\mathbb{C}}$ with some other hermitian symmetric space, i.e., a symmetric space with a 'nice' complex structure. To start down either of these paths, see [1].

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