DOMAINS OF DISCONTINUITY FOR ANOSOV REPRESENTATIONS

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This article is a summary of the content in Section 7 to 10 of [3]. Let Γ be a word-hyperbolic group. The goal is to construct domains of discontinuity for a given Anosov representation ρ from Γ to a semisimple Lie group G. More specifically, we want to construct an open, Γ-invariant subset Ω in a compact G-space on which Γ acts properly discontinuously and cocompactly.

The general strategy for this construction is the following. In Section 2, we first construct domains of discontinuity for representations of Γ into automorphism groups of non-degenerate sesquilinear forms that are Anosov with respect to a stabilizer of an isotropic line or a maximal isotropic subspace. Using this, we can construct domains of discontinuity for Anosov representations of Γ into $SL(n, K)$, where $K = \mathbb{R}$ or $\mathbb{C}$. This is described in Section 3. In Section 4 we demonstrate how to generalize this construction for Anosov representations into a semisimple Lie group. Finally, we end the article by mentioning in Section 5 some structural results about these domains of discontinuity.

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1. Basics

We will start by giving a quick review of the general structure theory of semisimple Lie groups. The main purpose of this review is to establish notation that will be used in the rest of this article. For more details, one can refer to Chapter 2 of [2]. A brief description of this structure theory is also given in Section 3 of [3]. Let G be a semisimple Lie group and let $\mathfrak{g}$ be its Lie algebra. Choose a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$. This gives us a root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Lambda} \mathfrak{g}_\alpha,$$

where Λ is the set of roots for $\mathfrak{a}$. By choosing a Weyl chamber of this root space decomposition to be the positive Weyl chamber, we get a partition of Λ into the set of positive roots, denoted Λ^+, and the set of negative roots, denoted Λ^-. This also gives us a set of simple roots, denoted $\Delta \subset \Lambda^+$. Define

$$A := \exp(\mathfrak{a}),$$
$$\mathfrak{n} := \bigoplus_{\alpha \in \Lambda^+} \mathfrak{g}_\alpha$$
$$N := \exp(\mathfrak{n}),$$

and let K be the maximal compact subgroup of G with Lie algebra $\mathfrak{k}$.

Now, given any subset $\Theta \subset \Delta$, we can define a parabolic subgroup $P_\Theta := M_\Theta AN$, where M_Θ is the subgroup of K that fixes every element (via the adjoint representation) in $\bigcap_{\alpha\in\Theta}\ker(\alpha)$. The Lie algebra of P_Θ has a decomposition

$$\mathfrak{p}_\Theta = \bigoplus_{\alpha\in\Lambda^+\cup\Lambda_\Theta\cup\{0\}} \mathfrak{g}_\alpha,$$

where Λ_Θ is the set of roots that lie in the $\mathbb{R}$-span of the simple roots in Θ. It is known that any parabolic subgroup of G is conjugate to a parabolic subgroup P_Θ for some Θ.

2. Automorphism groups of non-degenerate sesquilinear forms

We will now describe the construction of domains of discontinuity for a very special type of representation $\rho : \Gamma \to G$.

Notation 2.1. Let (V, F) be a vector space over either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, equipped with a non-degenerate quadratic form F such that

(1) F is either an indefinite symmetric form or a skew-symmetric form when V is over $\mathbb{R}$,
(2) F is either a symmetric form, a skew-symmetric form or an indefinite Hermitian form when V is over $\mathbb{C}$,
(3) F is either an indefinite Hermitian form or a skew-hermitian form when V is over $\mathbb{H}$.

Denote the group of automorphisms of (V, F) by $G_F(V)$. Also, let $\mathcal{F}_0$ be the set of isotropic lines in V and let $\mathcal{F}_1$ be the set of maximal isotropic subspaces in V.

One can check that $G_F(V)$ acts transitively on both $\mathcal{F}_0$ and $\mathcal{F}_1$. Moreover, the stabilizers of isotropic lines and maximal isotropic subspaces in $G_F(V)$ are parabolic subgroups. In fact, we can describe what these parabolic subgroups are. The isotropic line stabilizers are conjugate to the parabolic subgroup $P_{\Delta\setminus\alpha_0}$ for some simple root α_0, and the parabolic subgroup that stabilizes a maximal isotropic subspace is conjugate to $P_{\Delta\setminus\alpha_1}$ for some simple root α_1. The roots α_0 and α_1 can be explicitly described, but we will not do so here. (See Section 7.2 of [3].) To simplify notation, we will denote $P_{\Delta\setminus\alpha_i}$ as Q_i. Using the orbit-stabilizer theorem, we can thus identify $\mathcal{F}_i$ with G/Q_i. In this section, we will only consider representations $\rho : \Gamma \to G$ that are Q_i-Anosov for some $i = 0, 1$.

Exercise 2.2. Work out what α_0 and α_1 are when G is $O(2, 3)$, $O(3, 3)$ and $Sp(4, \mathbb{R})$.

Let $\mathcal{F}_{01} := \{(D, P) \in \mathcal{F}_0 \times \mathcal{F}_1 : D \subset P\}$, and let $\pi_i : \mathcal{F}_{01} \to \mathcal{F}_i$ be the obvious projection. For any subset $A \subset \mathcal{F}_i$, let $K_A := \pi_{1-i}(\pi_i^{-1}(A))$, and observe that if A is closed in $\mathcal{F}_i$, then K_A is closed in $\mathcal{F}_{1-i}$. In particular, if $\rho : \Gamma \to G$ is Q_i-Anosov, then we have a Γ-invariant Anosov map $\xi : \partial_\infty\Gamma \to \mathcal{F}_i$ whose image is closed in $\mathcal{F}_i$, so $K_{\xi(\partial_\infty\Gamma)}$ is closed in $\mathcal{F}_{1-i}$. Let $\Omega_\rho := \mathcal{F}_{1-i} \setminus K_{\xi(\partial_\infty\Gamma)}$, and observe that Ω_ρ is an open, Γ-invariant subset of $\mathcal{F}_{1-i}$. In [3], Guichard and Wienhard proved the following theorem.

Theorem 2.3. *(Theorem 8.6 of* [3]*) Let $\rho : \Gamma \to G$ be a Q_i-Anosov representation. If Ω_ρ is nonempty, then Γ acts properly discontinuously and cocompactly on Ω_ρ.*

We will now give a vague description of their proof. To prove the proper discontinuity of the Γ action, they showed that the Q_i-Anosovness of ρ implies that its image satisfies some proximality conditions, which gives us some control of the Γ action on its limit set in $\mathcal{F}_i$. This then gives us enough control of the Γ action on $K_{\xi(\partial_\infty\Gamma)}$ to prove that Γ acts properly discontinuously on the complement. For the cocompactness of the action, they considered the orientable double cover $\mathcal{F}_i^{or}$ of $\mathcal{F}_i$. By making some suitable arrangements, they could assume without loss of generality that Γ is torsion free and the codimension of $K^{or}_{\xi(\partial_\infty\Gamma)}$ in $\mathcal{F}^{or}_{1-i}$ is sufficiently big. Hence, $\Gamma\backslash\Omega_\rho^{or}$ is a connected orientable manifold. Proving the cocompactness of the action of Γ in Ω_ρ is equivalent to proving the compactness of $\Gamma\backslash\Omega_\rho^{or}$, which in turn is equivalent to proving that $H^0_c(\Gamma\backslash\Omega_\rho^{or})$ (cohomology with compact support and coefficients in $\mathbb{R}$) is non-zero. This can be done using Poincaré duality and long exact sequences of homology groups.

In the course of proving Theorem 2.3, several important properties of $K_{\xi(\partial_\infty\Gamma)}$ are used. The two most important ones are listed here.

Proposition 2.4. *(Proposition 8.1 of* [3]*) Let $\rho : \Gamma \to G$ be a Q_i-Anosov representation with associated Anosov map $\xi : \partial_\infty\Gamma \to \mathcal{F}_i$. Then $\pi_{1-i} : \pi_i^{-1}(\xi(\partial_\infty\Gamma)) \to K_{\xi(\partial_\infty\Gamma)}$ is a homeomorphism.*

This proposition is a consequence of the transversality property of the Anosov map ξ. Also, it follows from this proposition that $K_{\xi(\partial_\infty\Gamma)} \simeq \pi_i^{-1}(\xi(\partial_\infty\Gamma))$ is a locally trivial fiber bundle over $\xi(\partial_\infty\Gamma) \simeq \partial_\infty\Gamma$. The fiber over a point $t \in \partial_\infty\Gamma$ is the set of lines through the origin in $\xi(t)$ if $i = 1$, and is the set of maximal isotropic subspaces containing $\xi(t)$ if $i = 0$.

The next proposition computes the "codimension" of $\partial_\infty\Gamma$ in $\mathcal{F}_{1-i}$. Here, the dimensions of the topological spaces are cohomological dimensions for Čech cohomology.

Proposition 2.5. *(Proposition 8.3 of* [3]*) Let $\rho : \Gamma \to G$ be a Q_i-Anosov representation. Let $\mathrm{vcd}(\Gamma)$ be the virtual cohomological dimension of Γ and let $\delta := \dim(\mathcal{F}_{i-1}) - \dim(K_{\xi(\partial_\infty\Gamma)})$. Then*

- *If $G = O(p,q)$, $U(p,q)$ or $Sp(p,q)$ (with $0 < p \le q$), then $\delta = q - \mathrm{vcd}(\Gamma)$, $2q - \mathrm{vcd}(\Gamma)$ or $4q - \mathrm{vcd}(\Gamma)$ respectively.*
- *If $G = O(2n,\mathbb{C})$ or $O(2n-1,\mathbb{C})$, then $\delta = 2n - \mathrm{vcd}(\Gamma)$.*
- *If $G = Sp(2n,\mathbb{R})$ or $Sp(2n,\mathbb{C})$, then $\delta = n+1-\mathrm{vcd}(\Gamma)$ or $2n+2-\mathrm{vcd}(\Gamma)$ respectively.*
- *If $G = SO^*(2n)$, then $\delta = 4n-2-\mathrm{vcd}(\Gamma)$.*

In particular, if $\delta > 0$ then Ω_ρ is non-empty. To prove Proposition 2.5, we use Proposition 2.4 to get that $\dim(K_{\xi(\partial_\infty\Gamma)}) = \dim(M) + \dim(\partial_\infty\Gamma)$, where M is a fiber π_{1-i}. It is well-known result by Bestvina and Mess (see Corollary 1.4 of [1]) that $\dim(\partial_\infty\Gamma) = \mathrm{vcd}(\Gamma) - 1$. Thus, computing $\dim(M)$ in each of the cases will give us the statements in Propositon 2.5.

Exercise 2.6. Explicitly compute δ for a Q_0-Anosov representation of the fundamental group of a closed surface with genus at least 2 into $O(2,3)$.

3. General strategy and the $SL(n,K)$ case

We would like to use the domains of discontinuity constructed in Section 2 to construct domains of discontinuity for Anosov representations into a general semisimple

Lie group. The strategy to do so is the following. Let G be an arbitrary semisimple Lie group and let $\rho : \Gamma \to G$ be a representation. We want to find a group homomorphism $\phi : G \to G_F(V)$ such that

(I) $\phi \circ \rho : \Gamma \to G_F(V)$ is Q_0-Anosov (or Q_1-Anosov),
(II) There is a maximal isotropic subspace T (or an isotropic line D resp.) in V whose stabilizer in G is a proper subgroup of G.

Suppose that we have such a group homomorphism ϕ, then we can construct a domain of discontinuity for ρ in the G-space G/AN as follows. (We only describe the construction in the Q_0-Anosov case. The Q_1-Anosov case is similar.)

We can define the ϕ-equivariant injective map

$$\begin{aligned} \phi_0 :& G/\operatorname{Stab}_G(T) \to \mathcal{F}_0(V) = G_F(V)/Q_0 \\ & g \cdot \operatorname{Stab}_G(T) \mapsto \phi(g) \cdot D \end{aligned} \tag{3.1}$$

Moreover, condition (I) implies that we have an Anosov map $\xi_{\phi\circ\rho} : \partial_\infty\Gamma \to \mathcal{F}_0(V)$ for $\phi \circ \rho$. By Section 2, we have a domain of discontinuity $\Omega_{\phi\circ\rho}$ in $\mathcal{F}_1(V)$ for $\phi \circ \rho$. One can then check that $\Omega_{\rho,V,T} := \phi_1^{-1}(\Omega_{\phi\circ\rho})$ is an open subset of G/AN that is stabilized by Γ, and on which Γ acts properly discontinuously and cocompactly. This is the required domain of discontinuity.

Thus, we now only need to construct the homomorphism ϕ with the required properties. The basic tool used to verify if ϕ satisfies the condition (1) is the following proposition.

Proposition 3.1. *(Proposition 4.4 of* [3]*) Let $\phi : G \to G'$ be a Lie group homomorphism. Let $\mathfrak{a}^+$, $\mathfrak{a}'^+$ be the positive Weyl chambers for G and G' respectively, and let Δ, Δ' be the corresponding set of simple roots for G and G' respectively. Also, let W' be the Weyl group for G'. For any $\Theta' \subset \Delta'$, and let $W'_{\Theta'}$ be the Weyl group of $L'_{\Theta'}$. Let $\Theta \subset \Delta$ and suppose that there exist w' in W' and $\Theta' \subset \Delta'$ such that*

$$\phi_*(\overline{\mathfrak{a}^+} \setminus \bigcup_{\alpha\in\Theta} \ker(\alpha)) \subset w' \cdot W'_{\Theta'} \cdot (\overline{\mathfrak{a}'^+} \setminus \bigcup_{\alpha'\in\Theta'} \ker(\alpha')).$$

Then for any P^+_Θ-Anosov representation $\rho : \Gamma \to G$, the representation $\phi \circ \rho$ is ${P'}^+_{\Theta'}$-Anosov.

The idea behind the proof of this proposition is to convert the contraction properties in the definition of an Anosov representation into more Lie group theoretic conditions, so that it is easy to see whether the contraction is preserved when we compose the Anosov representation ρ with a group homomorphism ϕ. Guichard and Wienhard did this using what they call L-Cartan projections. For more details, see Section 3.3 of [3].

Using Proposition 3.1 and some standard representation techniques, we can now give explicit constructions (See Section 10 of [3]) of domains of discontinuity for all Anosov representations into $SL(n, K)$, where K is either $\mathbb{R}$ or $\mathbb{C}$. We will briefly describe how this is done.

From the definition of a P-Anosov representation, it is easy to see that every P-Anosov representation is also $P_{\Theta_1} \cap P_{\Theta_2}$-Anosov, where $P_{\Theta_1}, P_{\Theta_2}$ is some pair of maximal parabolic subgroups of G such that the opposite of P_{Θ_1} is conjugate to P_{Θ_2}, and both P_{Θ_1} and P_{Θ_2} contain N. If we choose $\mathfrak{a}$ to be the set of diagonal matrices in $\mathfrak{sl}(n, K)$ and $\mathfrak{n}$ to be the upper triangular matrices in $\mathfrak{sl}(n, K)$, then in the standard basis, there is some k such that P_{Θ_1} is the stabilizer of the subspace spanned by the first k basis vectors and P_{Θ_2} is the stabilizer of the subspace spanned by the first $n-k$ basis vectors. In this case, it is clear that the natural map $\phi : SL(n, K) \to O(\mathrm{End}(\bigwedge^k K^n), tr)$ satisfies condition (I), and we can check that it satisfies condition (II) using Proposition 3.1. Thus, we can construct domains of discontinuity for P-Anosov representations to $SL(n, K)$ for any proper parabolic subgroup P in $SL(n, K)$.

Exercise 3.2. Consider the case when $G = SL(5, \mathbb{R})$. List all the possible parabolic subgroups of the form $P_{\Theta_1} \cap P_{\Theta_2}$ as described above. Also, give explicit descriptions of the domains of discontinuity for $P_{\Theta_1} \cap P_{\Theta_2}$-Anosov representations for all possible $P_{\Theta_1} \cap P_{\Theta_2}$.

4. Generalization to other semisimple Lie groups

To further extend this to the case of a general semisimple Lie group G, we use the following proposition.

Proposition 4.1. *(Proposition 4.3 of* [3]*) Let $\phi : G \to SL(V)$ be an irreducible finite dimensional linear representation of G. Let $V = D \oplus H$ be a decomposition of V into a line and a hyperplane, and set $Q_0^+ = \mathrm{Stab}_{SL(V)}(D)$, $Q_0^- = \mathrm{Stab}_{SL(V)}(H)$. Suppose that $(P^+, P^-) = (\mathrm{Stab}_G(D), \mathrm{Stab}_G(H))$ is a pair of opposite parabolic subgroups. Then a representation $\rho : \Gamma \to G$ is (P^+, P^-)-Anosov if and only if $\phi \circ \rho : \Gamma \to SL(V)$ is (Q_0^+, Q_0^-)-Anosov.*

This is in fact a specialization of Proposition 3.1 to the case when ϕ is irreducible and $G' = SL(V)$ for some finite dimensional vector space V.

To use this proposition, we need to build, for any semisimple Lie group G and any proper parabolic subgroup $P \subset G$, an irreducible representation $\phi : G \to SL(V)$ such that $P = \mathrm{Stab}_G(D)$ for some line D in V. This is not difficult to do using some standard representation theory. For instance, one can simply take $V = \bigwedge^p \mathfrak{g}$, where $\mathfrak{g}$ is the lie algebra of G and p is the dimension of the Lie algebra $\mathfrak{p}$ of P.

Now, given a P-Anosov representation $\rho : \Gamma \to G$, where G is a semisimple Lie group and P is a proper parabolic subgroup of G, we first construct an irreducible representation $\phi_1 : G \to SL(V)$ satisfying the conditions in Proposition 4.1. Then $\phi_1 \circ \rho$ is Anosov with respect to a line stabilizer, which is a proper parabolic subgroup of $SL(V)$. By the last paragraph of Section 3, we can then construct a representation $\phi_2 : SL(V) \to O(\mathrm{End}(K^n), tr)$ so that $\phi_2 \circ \phi_1 \circ \rho$ is Anosov with respect to the stabilizer of an isotropic line in $(\mathrm{End}(K^n), tr)$. Here, $K = \mathbb{C}$ if V is a complex vector space and $K = \mathbb{R}$ if V is a real vector space. Then one can check that $\phi = \phi_2 \circ \phi_1$ satisfies conditions (I) and (II), so we can use ϕ in the general strategy given at the start of Section 3 to obtain a domain of discontinuity for ρ.

5. Structure of the domain of discontinuity

Guichard and Wienhard also managed to obtain several results about the structure of these domains of discontinuity. We will end this article by mentioning two such results.

The first is an attempt to generalize Proposition 2.5 to the setting when G is an arbitrary semisimple Lie group. This is difficult to do in general, but in the cases where the virtual cohomological dimension of Γ is one or two, there are some concrete conditions to determine when the domain of discontinuity we constructed is nonempty. The main theorem in this direction is the following.

Theorem 5.1. *(Theorem 9.10 of* [3]*) Let $\rho : \Gamma \to G$ be a P-Anosov representation for which we can construct a domain of discontinuity, $\Omega_{\rho,V,T} \subset G/AN$.*

(1) If $\mathrm{vcd}(\Gamma) = 1$, then $\Omega_{\rho,V,T}$ is non-empty.
(2) If $\mathrm{vcd}(\Gamma) = 2$ and P contains every factor of G that is locally isomorphic to $SL(2,\mathbb{R})$, then $\Omega_{\rho,V,T}$ is nonempty.

The second result is about the topology of the quotient $\Gamma\backslash\Omega_{\rho,V,T}$. In general, it is not easy to determine the topology of $\Gamma\backslash\Omega_{\rho,V,T}$, but we have the following theorem.

Theorem 5.2. *(Theorem 9.12 of* [3]*) Let $\rho : \Gamma \to G$ be a P-Anosov representation for which we can construct a domain of discontinuity $\Omega_{\rho,V,T} \subset G/AN$. Let $\mathrm{Hom}_{P\text{-}Anosov}(\Gamma, G)$ be the space of P-Anosov homomorphisms from Γ to G. Then there exists an open neighborhood U of ρ in $\mathrm{Hom}_{P\text{-}Anosov}(\Gamma, G)$ such that*

$$\Gamma\backslash \bigcup_{\rho' \in U} \Omega_{\rho',V,T} \simeq U \times (\Gamma\backslash\Omega_{\rho,V,T})$$

as bundles over U.

In particular, the topology of $\Gamma\backslash\Omega_{\rho,V,T}$ depends only on the connected component of $\mathrm{Hom}_{\text{P-Anosov}}(\Gamma, G)$ containing ρ.

References

[1] M. Bestvina, G. Mess *The boundary of negatively curved groups* J. Amer. Math. Soc. 4 (1991), no.3, 469-481.
[2] P. Eberlein *Geometry of Non-positively Curved Manifolds* Chicago Lectures in Mathematics, 1997, ISBN-10: 0226181987.
[3] O. Guichard, A. Wienhard *Anosov representations: Domains of discontinuity and applications* Invent. Math. 190 (2012), no.2, 357-438.