On the Hölder regularity of convex divisible domains

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Abstract

We demonstrate the equality, for divisible convex domains, of the Hölder regularity of the boundary of the domain and the boundary of the dual domain.

0 Introduction

A convex open domain Ω in projective space $\mathbb{P}^n(\mathbb{R})$ is called *divisible* if there exists a subgroup of $\mathrm{PGL}_{n+1}(\mathbb{R})$ acting properly on the domain Ω with a compact quotient (this terminology is due to Vey [12]).

Here, we are interested only in the case of Ω being *sharp* (fr. *saillant*), which means that there exists a hyperplane H of $\mathbb{P}^n(\mathbb{R})$ such that $H \cap \overline{\Omega} = \emptyset$ and moreover the domain Ω is *strictly convex*, i.e. for all hyperplanes H not intersecting Ω, the closure $\overline{\Omega}$ contains at most one point of H. Such a domain Ω is called a *strictly convex divisible domain*. For an introduction and most complete exposition of the properties of convex divisible domains, see articles by Benoist [4, 5].

The first series of examples comes from convex projective structures on a compact surface of genus g greater than two (a manifold is called convex if each homotopy class with fixed endpoints has at most one geodesic). The universal cover of a surface is then identified with a strictly convex divisible domain in projective space (see Theorem 3 of [9]). The most powerful description of projective surfaces is given by Goldman and Choi [8, 6].

Other examples are given by compact manifolds of constant curvature -1, in which case the domain Ω in $\mathbb{P}^n(\mathbb{R})$ is an ellipsoid, that is, the image in $\mathbb{P}^n(\mathbb{R})$ of the future cone of the quadratic form $\mathbf{q}$ of signature $(1, n)$, and the group Γ is a cocompact subgroup of $\mathrm{SO}(\mathbf{q})$. The deformations of the projective structure of the quotient manifold $M = \Gamma\backslash\Omega$ provide non-trivial examples of strictly convex divisible domains; see for example Corollary 2.10 of [5] and Remark 1.3 of [2].

In general, the boundary $\partial\Omega$ does not have the regularity of ellipsoids. However, it is α-Hölder for some number $1 < \alpha \leq 2$ (Proposition 4.6 of [5]). The supremum of these numbers α is called the regularity of Ω. However, α_Ω is equal to 2 only when the domain Ω is an ellipsoid (Proposition 6.1 of [5]).

The point of this text is to demonstrate that for a strictly convex divisible domain Ω the regularity of Ω and of the dual convex domain Ω^* coincide (Theorem 4), which responds to a question posed in [5]. The key point is to demonstrate that this regularity can be found using the eigenvalues of the elements of Γ (Theorem 22).

1 Convex Divisible Domains

This section repeats the definition of convex divisible domains and those of their properties that will be useful to us below and for the main theorem.

Denote by V a real vector space of dimension $n+1$.

Definition 1. Let Ω be a domain in $\mathbb{P}(V)$.

The domain Ω is called *convex and divisible* if it has the following properties:

1. Ω is a convex (that is, the intersection of Ω with any projective line is connected) and sharp.
2. There exists a discrete subgroup Γ of $\mathrm{PGL}(V)$ leaving Ω invariant such that the quotient $\Gamma\backslash\Omega$ is compact.

Remarks

- Throughout this text, we are interested in only the convex divisible domains that are furthermore strictly convex, i.e. any hyperplane H that does not intersect Ω intersects $\overline{\Omega}$ in at most one point.
- Up to passing to a subgroup of finite index, the group Γ can be assumed to be *torsion-free* (it is a classic result that the group Γ is finitely generated because it is discrete and acts cocompactly on Ω. This allows the application of Selberg's lemma to produce a torsion-free subgroup of finite index [11]), and that the quotient $M = \Gamma\backslash\Omega$ is a compact *manifold*.

1.1 Dual Convex Domain

Definition 2. The dual Ω^* of Ω is:

$$\Omega^* = \{ \, [f] \in \mathbb{P}(V^*) \mid f(x) \neq 0 \text{ for all } [x] \text{ in } \overline{\Omega} \, \}$$

If the domain Ω is sharp, then the dual Ω^* is a non-empty convex domain.

Proposition 3. *The following are equivalent:*

1. *Ω is a strictly convex divisible domain.*
2. *Ω^* is a strictly convex divisible domain.*

Proof. By definition the domain Ω is sharp, so the dual Ω^* is non-empty and open. Furthermore, it is clear that if the group Γ leaves Ω invariant, then the group ${}^t\Gamma$ leaves Ω^* invariant.

The equivalence between the compactness of $\Gamma\backslash\Omega$ and that of ${}^t\Gamma\backslash\Omega^*$ is shown by Lemma 2.8 of [5]. Thus, Ω is divisible if and only if Ω^* is divisible.

Furthermore, a convex divisible domain Ω is strictly convex if and only if the group Γ is Gromov hyperbolic (Proposition 2.5 and Fact 2.4 of [5]). This shows the equivalence between the strict convexity of the convex divisible domain Ω and that of its dual. □

Since the domain Ω^* is strictly convex, the boundary $\partial\Omega^*$ is of type C^1.

A strictly convex divisible domain has maximal regularity α_Ω, which we will define precisely in Section 3. The main theorem that we would like to show is the following.

Theorem 4. *Let Ω be a strictly convex divisible domain in $\mathbb{P}(V)$, with the dual Ω^* also strictly convex and divisible by Proposition 3. The regularities of Ω and Ω^* are equal:*

$$\alpha_\Omega = \alpha_{\Omega^*}.$$

Theorem 33 will make this theorem more precise and complete, notably through the fact that the limit regularity is attained and the results concerning the geodesic flow on the quotient $\Gamma\backslash\Omega$.

2 Hilbert Distance

Based on the definition of distance in hyperbolic space $\mathbb{H}^n$, the following distance is defined on a convex domain Ω:

$$\text{for } x \text{ and } y \text{ in } \Omega, \quad \begin{cases} d_\Omega(x,y) = 0 & \text{if } x = y, \\ d_\Omega(x,y) = |\log(a,b;x,y)| & \text{if } x \neq y \end{cases}$$

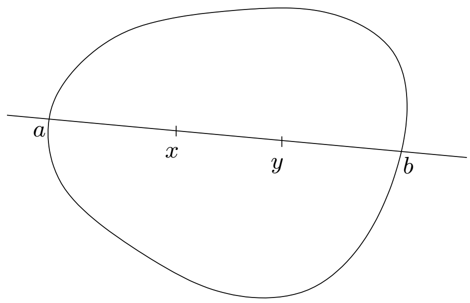

Figure 1: The Hilbert metric

where a and b are the two points in the intersection of the projective line (x,y) with the boundary $\partial\Omega$ (Figure 1), and $(a,b;x,y)$ designates the cross-ratio of the four points a,b,x,y:

$$(a,b;x,y) = \frac{x-a}{x-b}\frac{y-b}{y-a}$$

The function d_Ω satisfies the axioms of a distance.

This distance is a Finsler metric, defined by a continuous metric on the tangent fiber. This distance has an associated geodesic flow, which is proven here to be Anosov (Section 4). The regularity of the geodesic flow and that of the boundary $\partial\Omega$ are related, and we leave precise comments concerning this for Section 4.

We repeat the formulas for the metric and the geodesic flow. Let H be a hyperplane such that $\mathbb{P}(H)$ does not intersect $\overline{\Omega}$, so that Ω can be identified with a convex relatively compact domain of H and the tangent bundle $T\Omega$ with $\Omega \times H$.

Let $\omega = (x,\xi)$ in $T\Omega$; there then exist two points in the boundary, $p^+(\omega)$ and $p^-(\omega)$, and two strictly positive numbers $\sigma^+(\omega)$ and $\sigma^-(\omega)$, see Figure 2, such that

$$\xi = \sigma^+(\omega)(p^+(\omega) - x),$$
$$\xi = \sigma^-(\omega)(x - p^-(\omega)).$$

The expressions for the norm of ω and the flow are then the following:

$$\|\omega\|_\Omega = \sigma^+(\omega) + \sigma^-(\omega)$$

$$\phi^t \cdot \omega = \left(x + \frac{e^t - 1}{\sigma^+(\omega)e^t + \sigma^-(\omega)}\xi, \frac{e^t}{(\sigma^+(\omega)e^t + \sigma^-(\omega))^2}\xi \right).$$

The group Γ acts by isometries on (Ω, d_Ω). In fact, the elements of Γ leave invariant the domain Ω and its boundary $\partial\Omega$, and the projective transformations preserve the cross-ratio.

3 α-Hölder Regularity and β-convexity

This section introduces the notions of α-Hölder regularity and β-convexity, and gives a characterization of β-convexity for functions. We also recall results concerning the regularity of the boundary of a strictly convex divisible domain.

4 α-Hölder Regularity

Definition 5. Let M be a hypersurface of class C^1 in $\mathbb{R}^n$. M is called *α-Hölder of class C^α* for a real number α, $1 < \alpha \leq 2$, if for all compact $K \subseteq \Omega$ there exists a constant C_K such that

$$d(x, T_yM) \leq C_K d(x,y)^\alpha \text{ for all } x \text{ and } y \text{ in } K.$$

Remarks

- The distances considered here are given by the norm on $\mathbb{R}^n$, and the tangent space T_yM is here implicitly considered an affine subspace of $\mathbb{R}^n$.
- In the case of the manifold M being the graph of a function f of class C^1, this definition is equivalent to (α-1)-Hölder continuity of the derivative of f. The function f is also called α-Hölder.

Here, the boundary $\partial\Omega$ is naturally a hypersurface of a vector space, because the closure of Ω is included in an affine patch of $\mathbb{P}(V)$. We talk of the C^α regularity of $\partial\Omega$, and the definition does not depend on the affine patch considered (Hölder regularity is in fact a local property).

We know that the boundary of strictly convex divisible domains is α-Hölder.

Proposition 6. *(Proposition 4.6 of [5]) Let Ω be a strictly convex divisible domain in $\mathbb{P}(V)$. There exists a real number $\alpha, 1 < \alpha \leq 2$ for which the boundary $\partial\Omega$ is of class C^α.*

We set

$$\alpha_\Omega = \sup\{\alpha \leq 2 \mid \partial\Omega \text{ is of class } C^\alpha\}.$$

By the proposition, α_Ω is strictly greater than one. One of the consequences of what we show is that the boundary of Ω is α_Ω-Hölder, that is to say that the limit value is attained.

4.1 β-convexity

Definition 7. Let M be a hypersurface of class C^1 of $\mathbb{R}^n$. M is called *β-convex* for some real number β more than or equal to two if, for all compact K in D, there exists a strictly positive constant C_K such that

$$d(x, T_yM) \geq C_K d(x,y)^\beta \text{ for all } x \text{ and } y \text{ in } K.$$

Remarks

- Again, this definition applies to the boundary $\partial\Omega$.

- If M is the graph of a C^1 convex function f, β-convexity of M is equivalent to β-convexity of f (see Defiition 10).

We will now connect this to Hölder regularity.

Lemma 8. *Let Ω be a strictly convex sharp domain $\mathbb{P}(V)$, Ω^* its dual in $\mathbb{P}(V^*)$, and α, β such that $1 < \alpha \leq 2 \leq \beta < \infty$, satisfying the equality*

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

Then the following are equivalent:

1. *$\partial\Omega$ is β-convex;*
2. *$\partial\Omega^*$ is C^α.*

Proof. The proof uses the following two points:

- if $\Omega \subset \Omega'$, then $\Omega'^* \subset \Omega^*$;
- (FALSE - Look up counterexample.) the dual of the domain

$$\Omega_1 = \{[x, y, 1] \in \mathbb{P}(\mathbb{R}^3) \mid |y| > |x|^\beta\}$$

  is

$$\Omega_1^* = \{[u, 1, v] \in \mathbb{P}((\mathbb{R}^3)^*) \mid |v| > |u|^\alpha\}.$$

□

This lemma combines with Proposition 6 to give the following proposition.

Proposition 9. *Let Ω be a strictly convex divisible domain. Then there exists a real number $\beta \geq 2$, for which $\partial\Omega$ is β-convex.*

We define

$$\beta_\Omega = \inf\{\beta \geq 2 \mid \partial\Omega \text{ is } \beta\text{-convex}\}.$$

By Lemma 8 we then have the relation

$$\frac{1}{\alpha_{\Omega^*}} + \frac{1}{\beta_\Omega} = 1.$$

During the rest of this text, we will prefer β-convexity. However, it is in terms of α-Hölder regularity of the boundary and the regularity of the geodesic flow that one should think of the results (cf. Section 4 and Theorem 22). We will show the following, equivalent to Theorem 4:

$$\beta_\Omega = \beta^*_\Omega,$$

where Ω is a strictly convex divisible domain and Ω^* is its dual.

4.2 β-convex functions

This paragraph provides a characterization (and a definition) of β-convex functions, which will be useful in Section 5.

Definition 10. Let U be a convex domain in $\mathbb{R}^n$, $f : U \to \mathbb{R}$ a convex C^1 function, β a real number greater than or equal to two. The following two propositions are then equivalent:

1. There exists a strictly positive constant C_1 such that, for all x and y in U,

$$|f(y) - f(x) - Df_x(y-x)| \geq C_1 \|y-x\|^\beta;$$

2. There exists a strictly positive constant C_2 such that, for all x and y in U,

$$\left| f(y) + f(x) - 2f\left(\frac{x+y}{2}\right) \right| \geq C_2 \|y-x\|^\beta.$$

The function f is called β-convex if it satisfies these two conditions.

Proof. The proof of (1) $\Rightarrow$ (2) is easy.

For the other direction, we may suppose $x = 0$ and $Df_0 = 0$. Because the function f is convex, for all $y \in U$, we have

$$f(0) + f(y) - 2f\left(\frac{y}{2}\right) = \left| f(0) + f(y) - 2f\left(\frac{y}{2}\right) \right| \geq C_2 \|y\|^\beta.$$

$$f(y) - f(0) \geq C_2 \|y\|^\beta + 2\left(f\left(\frac{y}{2}\right) - f(0) \right).$$

By induction, for all $n \in \mathbb{N}^*$

$$f(y) - f(0) \geq C_2 \sum_{k=0}^{n-1} 2^{k(1-\beta)} \|y\|^\beta + 2^n \left(f\left(\frac{y}{2^n}\right) - f(0) \right)$$

The right side has a limit as n goes to $+\infty$

$$\lim_{n\to\infty} 2^n \left(f\left(\frac{y}{2^n}\right) - f(0) \right) = Df_0(y) = 0.$$

so

$$f(y) - f(0) - Df_0(y)| = f(y) - f(0) \geq \frac{C_2}{1 - 2^{1-\beta}} \|y\|^\beta.$$

The constants satisfy $C_1 = C_2/(1 - 2^{1-\beta}) > 0$. □

5 Geodesic flow

We repeat here, without proof, the regularities of the geodesic flow ϕ^t on the quotient manifold $M = \Gamma \backslash \Omega$, where Ω is a strictly convex domain divided by a group Γ.

5.1 The Anosov property

Definition 11. Let W be a compact manifold with a Riemannian metric $\|\cdot\|_W$, and let ϕ^t be a flow on W defined by a vector field X.

The flow ϕ^t is called *Anosov* if there exists a continuous decomposition of the tangent bundle $TW = E^+ \oplus \mathbb{R}X \oplus E^-$ fixed by the flow, and if there exist strictly positive constants a and A such that, for all $t > 0, v^+ \in E^+, v^- \in E^-$, we have:

$$\|D\phi^{-t}(v^+)\|_W \leq Ae^{-at}\|v^+\|_W \text{ and } \|D\phi^t(v^-)\|_W \leq Ae^{-at}\|v^-\|_W. \tag{a}$$

The distribution E^+ is called unstable, and the distribution E^- is the stable distribution.

The supremum of the numbers a satisfying the inequalities (a) is denoted by a_ϕ.

If the manifold W is compact, there exist strictly positive constants b and B such that, for all $t > 0, v^+ \in E^+, v^- \in E^-$, we have

$$\|D\phi^{-t}(v^+)\|_W \geq Be^{-bt}\|v^+\|_W \text{ and } \|D\phi^t(v^-)\|_W \geq Be^{-bt}\|v^-\|_W. \tag{b}$$

The infimum of these numbers b is denoted by b_ϕ.

5.2 Stable and unstable distributions

Let Ω be a strictly convex divisible domain and $M = \Gamma\backslash\Omega$ the quotient manifold.

In this case, the distributions E^+ and E^- in T^1M for which the geodesic flow ϕ^t is Anosov (see the following Proposition 12), are in fact explicit. Furthermore, there are lifts $\widetilde{E}^+$ and $\widetilde{E}^-$ of these distributions in the unit tangent bundle $T^1\Omega$ that are formulated even more simply (Section 3.2.5 of [5]).

Let H be a hyperplane not intersecting $\overline{\Omega}$. The domain Ω can be identified with a domain in H, the tangent bundle $T\Omega$ with $\Omega \times H$, and, for all $\omega = (x, \xi)$ in $T\Omega$, the tangent space $T_\omega(T\Omega)$ with $H \times H$. We also use $p^+(\omega), p^-(\omega), \sigma^+(\omega), \sigma^-(\omega)$ here as in Section 2. We will also call H_ω^+ and H_ω^- the hyperplanes tangent to $\partial\Omega$ at the points $p^+(\omega)$ and $p^-(\omega)$, I_ω their intersection, and finally H_ω the hyperplane containing x and I_ω (see Figure 3).

The unstable and stable distributions are then the following:

$$\widetilde{E}_\omega^+ = \{v^+ \in T_\omega(T^1\Omega) \mid v^+ = (y, \sigma^-(\omega)y) \text{ for } y \in T_xH_\omega\},$$
$$\widetilde{E}_\omega^- = \{v^- \in T_\omega(T^1\Omega) \mid v^- = (y, -\sigma^+(\omega)y) \text{ for } y \in T_xH_\omega\}.$$

Furthermore, the action of the flow on these distributions admits the following geometric description (Section 3.2.6 of [5]). Let $\omega = (x, \xi)$ in $T^1\Omega$, $\omega_t = \phi^t \cdot \omega = (x_t, \xi_t)$ the geodesic passing through ω, and let $v^- = (y, -\sigma^+(\omega)y)$ be a point in $\widetilde{E}_\omega^-$. We then have $D\phi^t(v^-) = (y_t, -\sigma^+(\omega_t)y_t)$. The element y_t is the unique vector in $T_{x_t}H_{\omega_t}$ such that the three points $x + y, x_t + y_t$ and $p^+(\omega)$ are collinear.

5.3 The geodesic flow is Anosov

Proposition 12. *(Lemma 3.6 and Proposition 4.6 of [5]) Let Ω be a strictly convex divisible domain, and Γ a group (assumed torsion-free) such that the quotient $M = \Gamma\backslash\Omega$ is a compact manifold.*

The geodesic flow ϕ^t on the unit tangent bundle T^1M is then Anosov. The stable and unstable distributions E^- and E^+ are as in the previous paragraph. We furthermore have the following equivalences:

- *the boundary $\partial\Omega$ is α-Hölder $\Leftrightarrow$ the distribution E^- is $(\alpha - 1)$-Hölder;*

- *if $\alpha > 1$, the boundary is α-Hölder $\Leftrightarrow$ the inequalities (a) are true for $a = 1 - (1/\alpha)$;*
- *the boundary is β-convex $\Leftrightarrow$ the inequalities (b) are true for $b = 1 - (1/\beta)$.*

In particular, we have the equalities:

$$a_\phi = 1 - \frac{1}{\alpha_\Omega} \text{ and } b_\phi = 1 - \frac{1}{\beta_\Omega}.$$

One of the consequences of Theorem 22 will be that the limit values a_ϕ and b_ϕ are attained and that $a_\phi + b_\phi = 1$.

5.4 The Anosov properties on closed geodesics

Let W be a compact manifold acted on by Anosov flow ϕ^t.

We will denote by a'_ϕ the supremum of the numbers a such that the inequalities (a) are satisfied restricted to the periodic orbits of the flow on W. In other words, for each $a < a'_\phi$ and for each closed orbit $\sigma \subset W$, there exists a constant $A = A(a, \sigma)$ such that (a) is valid for all for all $t > 0, v^+$ in $E^+_{|\sigma}, v^-$ in $E^-_{|\sigma}$. The constant A does not need to be constant among all these orbits.

Perhaps the most striking consequence of the main Theorem 22 is the following corollary.

Corollary 13. *With the hypotheses of the previous proposition and the above notation, we then have the equality:*

$$a'_\phi = a_\phi.$$

6 β - convexity and Hilbert distance

Since the Hilbert metric is defined by points of $\partial\Omega$, it is natural to compare the properties of $\partial\Omega$ and those of the distance d_Ω. That is what we do in this section in connecting the β-convexity of $\partial\Omega$ and the "shape" of the balls of the domain Ω in the Hilbert metric.

In this section, Ω denotes a sharp strictly convex domain of $\mathbb{P}(V)$ (here Ω is not necessarily assumed to be divisible).

6.1 Shape of the balls

For this section, we fix a strictly positive number r_0. We are interested in the shape of the balls in the Hilbert metric. If x_0 belongs to Ω,

$$B_\Omega(x_0, r_0) := \{x \in \Omega \mid d_\Omega(x_0, x) \leq r_0\}.$$

To quantify this idea, let δ be a distance defining the topology on the closure $\overline{\Omega}$ and define, for a β more than or equal to two,

$$\begin{aligned}
\phi^\delta_\beta : \Omega &\longrightarrow \mathbb{R} \\
x_0 \longmapsto \phi^\delta_\beta(x_0) \quad &= \inf\left\{ \frac{\delta(x_0,y)}{\delta(x_0,x)^\beta} \;\middle|\; d_\Omega(x_0,y) = d_\Omega(x_0,x) = r_0 \right\} \\
&= \frac{\inf\{\delta(x_0,x) \mid d_\Omega(x_0,x) = r_0\}}{\sup\{\delta(x_0,y)^\beta \mid d_\Omega(x_0,y) = r_0\}}
\end{aligned}$$

and

$$\psi^\delta : [2,+\infty) \longrightarrow \mathbb{R}$$
$$\beta \longmapsto \psi^\delta(\beta) := \inf\{\phi^\delta_\beta(x_0) \mid x_0 \in \Omega\}.$$

Intuitively, we compare the length of the "minor axis" $a = \inf_{y\in\partial B_\Omega(y_0,r_0)} \delta(y_0,y)$ and the "major axis" $b = \sup_{x\in\partial B_\Omega(y_0,r_0)} \delta(y_0,x)$, of the balls when the point x_0 approaches the boundary $\partial\Omega$ and, more precisely, if the ratio a/b^β stays small.

These functions depend on the chosen distance in the following manner.

Lemma 14. *Let δ and δ' be two equivalent distances on $\overline{\Omega}$. Then for all x_0 in Ω,*

$$\frac{1}{C^{1+\beta}}\phi^\delta_\beta(x_0) \leq \phi^{\delta'}_\beta(x_0) \leq C^{1+\beta}\phi^\delta_\beta(x_0),$$

where the constant C is the constant of equivalence of the distances:

$$\textit{for all } x \textit{ and } y \textit{ in } \overline{\Omega}, \frac{1}{C}\delta(x,y) \leq \delta'(x,y) \leq C\delta(x,y).$$

The proof is direct. This lemma allows us to adapt the distances on $\overline{\Omega}$ to the most useful choice at each step, and shows that the statement "$\psi^\delta(\beta)$ is strictly positive" does not depend on the distance δ.

We now describe the distances on $\overline{\Omega}$ that we will use. Let H be a hyperplane in V such that the intersection of $\mathbb{P}(H)$ and Ω is empty. The closure $\overline{\Omega}$ is then identified with a compact subset of H and is endowed with the metric δ_H induced by the norm $\|\cdot\|_H$ on H. It is easy to see that the equivalence class of the metric thus defined depends neither on $\|\cdot\|_H$ nor on H.

6.2 β - convexity and the shape of the balls

The following proposition proves the connection with β-convexity.

Proposition 15. *Let Ω be a sharp convex domain in $\mathbb{P}^n(\mathbb{R})$ and β a number greater than or equal to two. If for (all) hyperplanes H, for which $\mathbb{P}(H)\cap\overline{\Omega} = \emptyset$, $\psi^{\delta_H}(\beta)$ is strictly positive, and the boundary $\partial\Omega$ is β-convex.*

Proof. The idea is approximately the following: allowing the ball to approach the boundary (Figure 4), the "minor diameter" a becomes equivalent to Ky, where K is a constant, and the "major diameter" b becomes equivalent to $K'x$. Furthermore, by hypothesis, there exists C such that $a \geq Cb^\beta$. This implies the existence of a number C' such that $y \geq C'x^\beta$. The rest of the proof makes this reasoning rigorous.

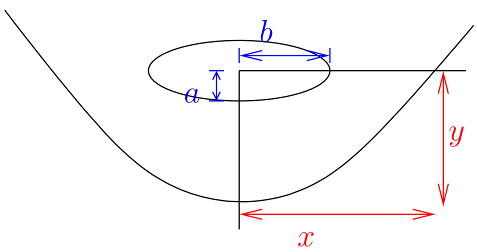

Figure 2: Idea of the proof.

The seek to apply point (2) of the definition of β-convex functions (Definition 10).

We work in the affine patch defined by the hyperplane H. Let $\|\cdot\|_H$ be a Euclidean norm in H.

Letting p_0 be a point in $\partial\Omega$, it suffices to prove the β-convexity in a neighborhood of p_0 (the property is local). There exists a neighborhood U of p_0 in the tangent space $T_{p_0}\partial\Omega$, a neighborhood U' of p_0 in the orthogonal $T_{p_0}\partial\Omega^\perp \simeq \mathbb{R}$ and a function $f : U \to U'$ satisfying:

- $U \times U'$ is a neighborhood of p_0 in H;
- $\partial\Omega \cap (U \times U') = \{(x, f(x)) \mid x \in U\}$.

The function f is convex because the domain Ω is convex. It is C^1 because the boundary $\partial\Omega$ is C^1.

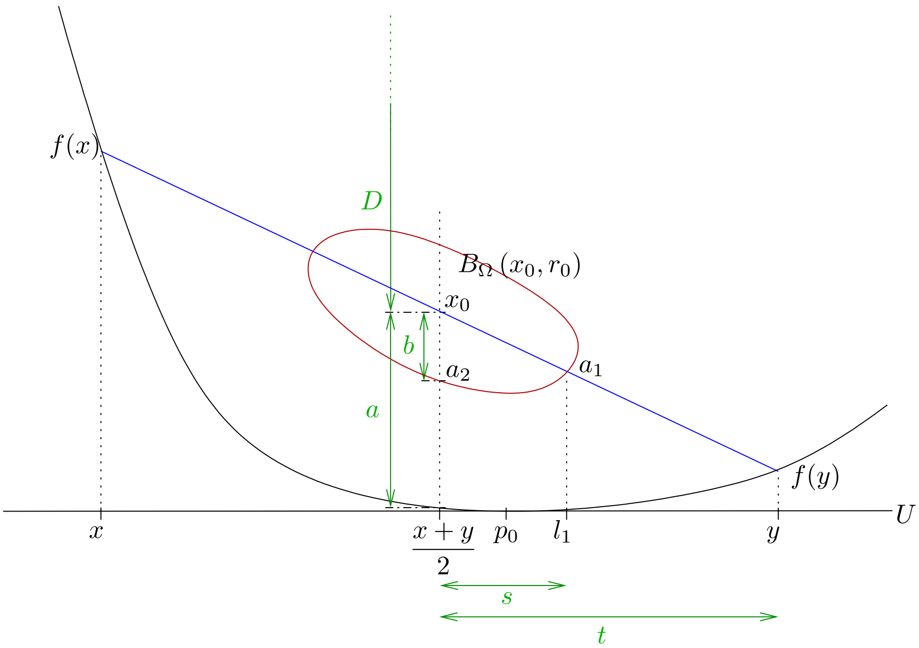

Figure 3: $\beta-$convexity of the boundary

Let x and y be in U.

We denote by x_0 the mean of $(x, f(x))$ and $(y, f(y))$, a_1 one of the points of intersection of the boundary $\partial B_\Omega(x_0, r_0)$ (r_0 being previously fixed) and the line passing through the two points $(x, f(x))$ and $(y, f(y))$. Let us denote, for example, by a_1 the point closer to y and by a_2 the point of the boundary $\partial B_\Omega(x_0, r_0)$ with horizontal coordinate (x+y)/2 and of smallest vertical coordinate.

The lengths a, b, s, t and D are indicated in Figure 5; D is the distance from x_0 to the second intersection point of the vertical line passing through x_0 and the boundary $\partial\Omega$.

Let us calculate $d_\Omega(x_0, a_1)$:

$$
\begin{aligned}
d_\Omega(x_0,a_1) &= |\log((x,f(x)),(y,f(y));x_0,a_1)| \\
&= \left|\log\left(x,y;\frac{x+y}{2},l_1\right)\right| \\
&= \left|\log\frac{t-s}{t}\frac{t}{t+s}\right|, \\
r_0 &= -\log\frac{t-s}{t+s}.
\end{aligned}
$$

The second equation above comes from the invariance of the cross-ratio under projections. We then have

$$t = s\frac{1+e^{-r_0}}{1-e^{-r_0}}.$$

and on the other hand,

$$t = \frac{\|x-y\|_H}{2}.$$

There then exists a constant C_1, independent of x and y, such that

$$\|x-y\|_H^\beta = C_1 s^\beta.$$

Furthermore, s is the smaller than $\|x_0 - a_1\|_H = \delta_H(x_0,a_1)$. We obtain

$$\|x-y\|_H^\beta \le C_1 \delta_H(x_0,a_1)^\beta. \tag{c}$$

Likewise, we calculate $d_\Omega(x_0,a_2)$:

$$
\begin{aligned}
d_\Omega(x_0,a_2) &= \left|\log\frac{b+D}{D}\frac{a}{a-b}\right| \\
\text{where} a &= \left|\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)\right|.
\end{aligned}
$$

Given the calculations, we have:

$$a = b\frac{e^{r_0}}{e^{r_0}-1-(b/D)}.$$

It is also easy to see that the distance D, which is a continuous function of the pair (x,y) belonging to $U\times U$, is uniformly bounded below on $U\times U$. Likewise, we can suppose by restricting U that the distance b is bounded above by an arbitrarily small constant. There then exists a strictly positive constant C_2 independent of x and y such that

$$\frac{e^{r_0}}{e^{r_0}-1-(b/D)} \ge C_2.$$

Since b is equal to $\delta_H(x_0,a_2)$, this shows that

$$\frac{f(x)+f(y)}{2} - f\left(\frac{(x+y)}{2}\right) \ge C_2\delta_H(x_0,a_2). \tag{d}$$

The inequalities (c) and (d) give ‘

$$\begin{aligned}\frac{1}{\|x-y\|_H^\beta}\left|\left(f(x)+f(y)-2f\left(\frac{x+y}{2}\right)\right)\right| &\geq \frac{C_2}{C_1}\frac{\delta_H(x_0,a_2)}{\delta_H(x_0,a_1)^\beta}\\ &\geq \frac{C_2}{C_1}\phi_\beta^{\delta_H}(x_0) \geq 2\frac{C_2}{C_1}\psi^{\delta_H}(\beta)=C\end{aligned}$$

and C is a strictly positive number. Finally, for all x and y in U, we have

$$\left|f(x)+f(y)-2f\left(\frac{x+y)}{2}\right)\right| \geq C\|x-y\|^\beta$$

That is to say, f is β-convex. □

RemarkWe similarly have the reciprocal proposition, i.e. β-convexity implies that $\psi^\delta(\beta)$ is strictly positive, but this will not be used in this paper.

7 (r,ε)-loxodromic elements and the Hilbert distance

The point of this part is to give certain properties of the loxodromic elements. We will use these properties in the proof of Theorem 22. We start by repeating the definitions of proximal and loxodromic elements.

7.1 Proximality

A fixed Euclidean structure $\langle\cdot,\cdot\rangle$ on V gives projective space $\mathbb{P}(V)$ a distance δ_0, associated to the norm $\|\cdot\|$ on V. If x and y are in $\mathbb{P}(V)$,

$$\delta_0(x,y) := \inf\{\|u-v\| \mid u\in x, v\in y, \|u\|=\|v\|=1\}.$$

If g is an element of $\mathrm{PGL}(V)$, g is *proximal* if g has a unique eigenvalue of largest modulus. This eigenvalue is real, and we denote by x^+ the associated eigen-line, and by $V_g^<$ the supplementary hyperplane fixed by g.

The action of g on $\mathbb{P}(V)$ then has a single attracting point x_g^+, and the basin of attraction is the complement of the hyperplane $X_g^< = \mathbb{P}(V_g^<)$.

Conversely, an element g that has an attracting fixed point in $\mathbb{P}(V)$ is proximal.

Definition 16. Let g be a proximal element and $r\geq\varepsilon>0$. We define

$$b_g^{\varepsilon,\delta_0} = B(x_g^+,\varepsilon) = \{x\in\mathbb{P}(V) \mid \delta_0(x,x_g^+)\leq\varepsilon\},$$

and

$$B_g^{\varepsilon,\delta_0} = \{x\in\mathbb{P}(V) \mid \delta_0(x,X_g^<)\geq\varepsilon\}.$$

We say g is (r,ε)-proximal if:

- $\delta_0(x_g^+,X_g^<)\geq 2r$;
- $g(B_g^{\varepsilon,\delta_0})\subset b_g^{\varepsilon,\delta_0}$;
- the restriction of g to $B_g^{\varepsilon,\delta^0}$ is ε-Lipschitz.

7.2 Loxodromic elements

An element $g \in \mathrm{PGL}(V)$ is called *loxodromic* (or $\mathbb{R}$-regular) if the eigenvalues of g, counted with multiplicity, have pairwise distinct norms. We denote them by $\lambda_1, \dots \lambda_{n+1}$ (they are necessarily real) such that $|\lambda_1| > \cdots > |\lambda_{n+1}|$, and let $v_1, \dots, v_{n+1}$ be associated eigenvectors.

For $i = 1, \dots, n+1$, we denote by V_i the i-th exterior power of V, $V_i = \bigwedge^i V$. Then the action of g on $\mathbb{P}(V_i)$ is proximal with attracting fixed point $x_i^+ = \mathbb{R}v_1 \wedge \cdots \wedge v_i$ and the basin of attraction is the compelement of $\mathbb{P}(V_i^<)$, where

$$V_i^< = \langle v_{k_1} \wedge \cdots \wedge v_{k_i} \mid k_1 < \cdots < k_i \text{ and } k_i > i \rangle.$$

Conversely, and element g for which the action is proximal on each $\mathbb{P}(V_i)$ is loxodromic.

The Euclidean structure on V induces a Euclidean structure on each V_i, $i = 1, \dots, n+1$ by the formula:

$$\langle v_1 \wedge \cdots \wedge v_i, w_1 \wedge \cdots \wedge w_i \rangle = \sum_{\sigma \in \mathfrak{S}_i} \varepsilon(\sigma) \langle v_1, w_{\sigma(1)} \rangle \cdots \langle v_i, w_{\sigma(i)} \rangle.$$

If $(e_i)_{i=1 \dots n+1}$ is an orthonormal basis for V, then an orthonormal basis of V_i is given by $(e_{k_1} \wedge \cdots \wedge e_{k_i})_{1 \leq k_1 < \cdots < k_i \leq n+1}$.

Definition 17. An element g in $\mathrm{PGL}(V)$ is called (r, ε)-loxodromic if the action of g on $\mathbb{P}(V_i)$ is (r, ε)-proximal for all i.

Below, the eigenvectors of g will be normalized, i.e. $\|v_1\| = \cdots \|v_{n+1}\| = 1$.

7.3 Loxodromic elements and the distance on $\mathbb{P}(V)$

For g loxodromic, we denote by $v_1, \dots, v_{n+1}$ the normalized eigenvectors of g, and by $\lambda_1, \dots \lambda_{n+1}$ the associated eigenvalues, ordered by decreasing modulus.

Let $\langle \cdot, \cdot \rangle_g$ be a dot product for which $v_1, \dots, v_{n+1}$ is an orthonormal basis, and δ_g the induced distance on $\mathbb{P}(V)$,

$$\delta_g(x, y) := \inf\{\|u - v\| \mid u \in x, v \in y, \|u\|_g = \|v\|_g = 1\}.$$

Since the projective space $\mathbb{P}(V)$ is compact, the distance δ_g is clearly equivalent to δ_0.

The point of this section is to show that, if g is (r, ε)-loxodromic, then the constants involved in the equivalence of the distances can be chosen independently of g.

We know there exists an orthonormal basis (e_i) of $(V, \langle \cdot, \cdot \rangle)$ and a lower-triangular matrix T with a strictly positive diagonal such that

$${}^t(v_1, \dots, v_{n+1}) = T^t(e_1, \dots e_{n+1}).$$

(This is the Schmidt orthonormalization process.) We denote by by $T(g)$ the lower-triangular matrix thus associated to g.

Proposition 18. *Let $r > 0$. There exists a compact K_r in $GL_{n+1}(\mathbb{R})$ such that if g is (r, ϵ)-loxodromic, then the matrix $T(g)$ belongs to the compact set K_r.*

Proof. Let $S(V)$ be a unit sphere in V with the norm $\|\cdot\|$. Let U be the domain in $S(V)^{n+1}$ of linearly independent $(n+1)$-tuples.

We denote by F_r the collection of $(n+1)$-tuples $(w_1, \dots w_{n+1})$ satisfying the following inequalities:

$$\delta_0(\mathbb{R}w_1, \mathbb{P}(W_1^<)) \geq 2r$$
$$\delta_0(\mathbb{R}w_1 \wedge w_2, \mathbb{P}(W_2^<)) \geq 2r$$
$$\dots$$
$$\delta_0(\mathbb{R}w_1 \wedge w_k, \mathbb{P}(W_k^<)) \geq 2r$$
$$\dots$$

where $W_k^< = \langle w_{i_1} \wedge \dots \wedge w_{i_k} \mid i_1 < \dots < i_k \rangle$, and δ_0 is the distance on $\mathbb{P}(V_i)$ induced by the Euclidean structure.

The set F_r is compact and in U. We denote by T the function

$$U \to GL_{n+1}(\mathbb{R})$$
$$(w_1, \dots, w_{n+1}) \mapsto T(w_1, \dots, w_{n+1})$$

where $T(w_1, \dots, w_{n+1})$ is the matrix given by the preceding Schmidt orthogonalization. This transformation T is continuous, so the image of F_r is a compact set K_r. □

Based on this proposition, we derive the corollary:

Corollary 19. *Let $r > 0$. There exists a strictly positive constant C_r such that for $g \in PGL(V)$, if g is (r, ε)-loxodromic, then*

$$\text{for all } x \text{ and } y \text{ in } \mathbb{P}(V), \frac{1}{C_r}\delta_g(x, y) \leq \delta_0(x, y) \leq C_r \delta_g(x, y).$$

δ_g is the distance associated with the norm $\| \cdot \|_g$ introduced earlier.

8 Equality of Hölder regularities

We will now prove the theorem announced in the first part of this paper (Theorem 4). We will in fact prove the stronger Theorem 22.

For the rest of this section, Ω designates a strictly convex divisible domain, and Γ the corresponding discrete subgroup of $PGL(V)$.

8.1 Notation.

We introduce some notation related to the group Γ.

Let $g \in PGL(V)$, $\lambda_1, \dots, \lambda_{n+1}$ the eigenvalues of g ordered by decreasing modulus. The logarithms of the norms of these eigenvalues are denoted by $l_1 \geq l_2 \geq \dots \geq l_{n+1}$.

If g is biproximal, that is to say g and g^{-1} are proximal, we set

$$\alpha_g := \frac{l_1 - l_{n+1}}{l_1 - l_n}, \quad \beta_g := \frac{l_1 - l_{n+1}}{l_1 - l_2}.$$

Furthermore, we define

$$\beta_\Gamma := \sup_{g \in \Gamma, g \text{ proximal}} \beta_g \text{ and } \alpha_\Gamma := \inf_{g \in \Gamma, g^{-1} \text{ proximal}} \alpha_g.$$

If g belongs to $\Gamma \backslash \{id\}$, we know that g is proximal (Proposition 5.1 of [5]), and so is g^{-1}, so

$$\frac{1}{\alpha_{g^{-1}}} + \frac{1}{\beta_g} = 1.$$

We then have

$$\frac{1}{\alpha_\Gamma} + \frac{1}{\beta_\Gamma} = 1. \tag{e}$$

We also have $\alpha_{({}^t g)} = \alpha_g$ and $\beta_{({}^t g)} = \beta_g$, and thus

$$\alpha_\Gamma = \alpha_{({}^t\Gamma)} \text{ and } \beta_\Gamma = \beta_{({}^t\Gamma)}. \tag{f}$$

8.2 Proximality and regularity

The numbers α_Γ and β_Γ are connected by the regularity of the boundary by the following inequalities:

Proposition 20. *(Corollary 5.3 of [5]) Let Ω a strictly convex divisible domain divided by group Γ. In the discussed notation, we have the inequalities*

$$1 < \alpha_\Omega \le \alpha_\Gamma \le 2 \le \beta_\Gamma \le \beta_\Omega < \infty.$$

We repeat the proof for the reader's convenience.

Proof. Let $g \neq \text{id}$ in Γ. We know that g is biproximal. We denote by x_g^+ and x_g^- the eigen-lines associated to the eigenvalues of the largest and smallest modulus, respectively. Let W_g be the g-invariant subspace supplementary to these lines.. The matrix of g in a basis adapted to the decomposition $x_g^+ \oplus W_g \oplus x_g^-$ is of the form

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & A_g & 0 \\ 0 & 0 & \lambda_{n+1} \end{bmatrix}.$$

The eigenvalues of g are $\lambda_1, \dots, \lambda_{n+1}$, and we denote by $l_1 \ge \cdots \ge l_{n+1}$ the logarithms of their norms. Furthermore, by Proposition 5.1 of [5] we know that x_g^+ belongs to the boundary $\partial\Omega$ and that the tangent hyperplane $T_{x_g^+}\partial\Omega$ is $V_g^> = x_g^+ \oplus W_g$.

In the affine patch defined by $V_g^< = W_g \oplus v_g^-$, we can place the point x_g^+ at the origin. The tangent hyperplane is then the subspace W_g. g acts on this coordinate patch by the matrix:

$$\begin{bmatrix} \lambda_1^{-1} A_g & 0 \\ 0 & \lambda_1^{-1}\lambda_{n+1} \end{bmatrix}$$

There exist elements p in the boundary $\partial\Omega$ such that:

$$\lim_{q\to\infty} \frac{1}{q} \log \delta_0(g^q \cdot p, x_g^+) = l_n - l_1 \text{ and } \lim_{q\to\infty} \frac{1}{q} \log \delta_0(g^q \cdot p, V_g^>) = l_{n+1} - l_1$$

Hölder regularity then shows that $\alpha_g \ge \alpha_\Omega$ and thus $\alpha_\Gamma \ge \alpha_\Omega$. The other inequality is shown in a similar manner. □

Geodesic flow and the eigenvalues of the elements of Γ are related.

Proposition 21. *Let Ω a strictly convex divisible domain and Γ a torsion-free group dividing Ω. Let furthermore a'_ϕ be the constant associated to the geodesic flow ϕ^t on the quotient $M = \Gamma\backslash\Omega$, as in Corollary 13.*

We then have the equation

$$a'_\phi = \frac{1}{\beta_\Gamma} = 1 - \frac{1}{\alpha_\Gamma}.$$

Proof. a'_ϕ is the contraction constant associated to the periodic geodesics. Let $a < 1/\beta_\Gamma$.

The closed geodesics of T^1M correspond to geodesics $(\omega_s)_{s\in\mathbb{R}}$ of $T^1\Omega$ such that there exists a non-trivial $g \in \Gamma$ such that $g \cdot (\omega_s)_{s\in\mathbb{R}} = (\omega_s)_{s\in\mathbb{R}}$.

We give the unit tangent bundle $T^1\Omega$ a Γ-invariant Riemannian metric $\|\cdot\|_{T^1\Omega}$. For such a geodesic $(\omega_s)_{s\in\mathbb{R}}$, we must show that there exists an A such that, for all s, for all $t > 0$, and for all v^- in $\widetilde{E}^-_{\omega_s}$:

$$\|(D\phi^t)_{\omega_s} v^-\|_{T^1\Omega} \leq Ae^{-at}\|v^-\|_{T^1\Omega}.$$

(The same proof works for $\widetilde{E}^+$.)

The logarithms of the norms of the eigenvalues of the element g, such that $g\cdot(\omega_s) = (\omega_s)$, are denoted by $l_1 \geq \cdots \geq l_{n+1}$. We then have for all s, $g \cdot \omega_s = \omega_{s+l_1-l_{n+1}}$, which comes from the formula for the Hilbert distance on Ω. By g-invariance, it suffices to prove the equation above for $s = 0$ and $t = q(l_1 - l_{n+1})$ with $q \in \mathbb{N}^*$. That is to say, there exists A such that for all $q \in \mathbb{N}^*$ and $v^- \in \widetilde{E}^-_{\omega_0}, t = q(l_1 - l_{n+1})$,

$$\|(D\phi^t)_{\omega_0} v^-\|_{T^1\Omega} \leq Ae^{-at}\|v^-\|_{T^1\Omega}.$$

Let H be a hyperplane such that $\mathbb{P}(H)$ does not intersect $\overline{\Omega}$, with the norm $\|\cdot\|_H$. As in Section 2, Ω is identified with a domain in H, the tangent bundle $T\Omega$ is identified with $\Omega \times H$ and, for ω in $T\Omega$, $T_\omega(T\Omega)$ with $H \times H$. We also use the notation of that section $(p^+, P^-, \sigma^+, \sigma^-)$.

For $\omega = \omega_0 = (x, \xi)$, we have $p^+(\omega) = x_g^+$ the eigen-line of g associated to the eigenvalue λ_1, and $p^-(\omega) = x_g^-$. The supplementary g-invariant subspace to these lines x_g^+ and x_g^- is denoted W_g. The hyperplane tangent to $\partial\Omega$ at x_g^+ is $x_g^+ \oplus W_g$, and that at x_g^- is $x_g^- \oplus W_g$.

Now let $v^- = (y, -\sigma^+(\omega)y)$ in $\widetilde{E}^-_\omega$, where y belongs to T_xH_ω (cf. Figure 3 where $I_\omega = W_g, p^+(\omega) = x_g^+$ and $p^-(\omega) = x_g^-$). Here H_ω is the hyperplane containing x and W_g. We then have $D\phi^t(v^-) = (y_t, -\sigma^+(\omega_t)y_t)$, and y_t is an element of $T_{x_t}H_{\omega_t}$ such that $x+y, x_t+y_t$ and x_g^+ are collinear (cf. Section 4.2). The hyperplane H_{ω_t} contains x_t and W_g.

According to Lemma 3.5 of [5], there exists a constant $C > 0$ such that

$$C^{-1}\|y_t\|_\Omega \leq \|D\phi^t(v^-)\|_{T^1\Omega} \leq C\|y_t\|_\Omega$$

($\|\cdot\|_\Omega$ is the metric on Ω defined by the Hilbert distance).

We need to find a constant A such that, for all y and all $t = q(l_1 - l_{n+1})$, we have $\|y_t\|_\Omega \leq Ae^{-at}\|y\|_\Omega$. By the definition of $\|\cdot\|_\Omega$, it suffices to obtain this inequality in terms of σ^+ and σ^-.

We have

$$\sigma^+(y_t) = \frac{\|y_t\|_H}{d_H(p^+(y_t), x_t)}.$$

$$\sigma^+(y) = \frac{\|y\|_H}{d_H(p^+(y), x)}.$$

Then,

$$\frac{\sigma^+(y_t)}{\sigma^+(y)} \leq \frac{\|y_t\|_H}{\|y\|_H} \frac{\max_{p\in H_\omega\cap\partial\Omega} d_H(p, x)}{\min_{p\in H_{\omega_t}\cap\partial\Omega} d_H(p, x_t)}$$

For the construction of y_t, we remark that, for all positive t, $\|y_t\|_H/\|y\|_H \leq ke^{-t}$, for a certain constant k. Furthermore, for $t = q(l_1 - l_{n+1})$, we remark that g^q sends $H_\omega \cap \partial\Omega$ to $H_{\omega_t} \cap \partial\Omega$ and we easily obtain the following inequality, for all $q > 0$ and $t = q(l_1 - l_{n+1})$.

$$\min_{p\in H_{\omega_t}\cap\partial\Omega} d_H(p,x_t) \geq Ke^{q(l_1-l_{n+1})} \min_{p\in H_\omega\cap\partial\Omega} d_H(p,x)$$

for a constant K (for example, using the affine chart in the previous proof).

This shows the existence of a constant A such that

$$\frac{\sigma^+(y_t)}{\sigma^+(y)} \leq Ae^{-t(1-1/\alpha_g)},$$

and since $a < 1 - 1/\alpha_g$, this will give the concluding inequality. We obtain that $a'_\phi \geq 1/\beta_\Gamma$.

Conversely, there exist vectors y such that, for all $t = q(l_1 - l_{n+1})$, we have

$$d_H(p^+(y_t), x_t) \geq Ke^{q(l_1-l_n)},$$
$$\|y_t\|_H \geq ke^{-t}\|y\|_H.$$

and the same calculation shows that $a'_\phi \leq 1 - 1/\alpha_g$ and we obtain the equality $a'_\phi = 1 - 1/\alpha_\Gamma$. □

8.3 Conclusion

The rest shows that the inequalities in Proposition 20 are in fact equalities, which will easily imply the desired result (Theorem 4) by equations (f).

Theorem 22. *Let Ω be a strictly convex divisible domain divided by group Γ. We furthermore denote by α_Ω and β_Ω the regularities of Ω. α_Γ and β_Γ are the constants associated with the group Γ in Section 7.1. a_ϕ and a'_ϕ are the constants related to the geodesic flow, defined in Section 4.4.*

Then:

1. *the boundary $\partial\Omega$ is α_Γ-Hölder; in particular $\alpha_\Omega = \alpha_\Gamma$;*
2. *the boundary $\partial\Omega$ is β_Γ-convex; and thus $\beta_\Omega = \beta_\Gamma$;*
3. *the contraction constants for the flow are equal: $a_\phi = a'_\phi = 1/\beta_\Omega$;*
4. *the regularities of Ω and its dual are equal: $\alpha_\Omega = \alpha_{\Omega^*}, \beta_\Omega = \beta_{\Omega^*}$.*

Lemma 8 and the inequalities (f) establish the equivalence of the points (1) and (2) of the theorem. Points (3) and (4) can be deduced from the previous two, by Propositions 12 and 21 and by the inequality (f).

Thus, we have to prove only the following proposition.

Proposition 23. *Let Ω a strictly convex domain divided by Γ. Let $\beta \geq \beta_\Gamma$. Then $\partial\Omega$ is β-convex.*

Proof. Suppose that the domain Ω is not an ellipsoid. We use the following fact:

there exists $r \geq \varepsilon > 0$ and a compact K in Ω, for which:
for all x in Ω, there exists y in K and g in Γ such that
g is (r,ε)-loxodromic and $x = g\cdot y$,

whose proof is delayed (Proposition 26).

Let $\beta > \beta_\Gamma$. By Proposition 15, it suffices to show that $\psi_{\delta_0}(\beta)$ is strictly positive, that is to say, that the function $\phi_\beta^{\delta_0}$, introduced in Section 5, is bounded below on the domain Ω.

This function is continuous and thus bounded below on the compact K by a strictly positive number c. Let C_r be the equivalence constant for the distances, given by Corollary 19, and $D = C_r^{1+\beta}$. Then, by Lemma 14, for all (r,ε)-loxodromic g and all x_0 in Ω,

$$\frac{1}{D}\phi_\beta^{\delta_g}(x_0) \le \phi_\beta^{\delta_0} \le D\phi_\beta^{\delta_g}(x_0).$$

and the function $\phi_\beta^{\delta_g}$ is bounded below on K by c/D (δ_g is the distance associated to g, introduced in Section 6).

Let g be (r,ε)-loxodromic, $\lambda_1,\dots,\lambda_{n+1}$ its eigenvalues ordered by decreasing modulus, $v_1,\dots,v_{n+1}$ the associated eigenvectors, and $l_1,\dots,l_{n+1}$ the logarithms of the norms of the eigenvalues.

The line $x_g^+ = \mathbb{R}v_1$ belongs to $\partial\Omega$ and is the attracting point of g in $\mathbb{P}(V)$. Likewise, $x_g^- = \mathbb{R}v_{n+1}$ is in $\partial\Omega$ and the hyperplane tangent to $\partial\Omega$ at x_g^- is $\mathbb{P}(V_g^<)$, where $V_g^< = \langle v_2,\dots,v_{n+1}\rangle$. The basin of attraction of g is the affine patch $B_g^+ = \mathbb{P}(V)\backslash\mathbb{P}(V_g^<)$:

$$\begin{aligned}\phi_g : V_g^< &\longrightarrow B_g^+\\ v &\longmapsto [v_1+v].\end{aligned}$$

The action of g is the following on this affine patch:

$$g * v_i := \phi_g^{-1}\circ g\circ\phi_g(v_i) = \frac{\lambda_i}{\lambda_1}v_i \text{ for all } i>1.$$

The vector space $V_g^<$ has a Euclidean norm $\|\cdot\|_g'$ for which $\{v_2,\dots,v_{n+1}\}$ is an orthonormal basis. In the manner discussed, we immediately get

$$\text{for all } v \text{ in } V_g^<, \left|\frac{\lambda_{n+1}}{\lambda_1}\right| \|v\|_g' \le \|g*v\|_g' \le \left|\frac{\lambda_2}{\lambda_1}\right| \|v\|_g',$$
$$e^{l_{n+1}-l_1} \le \frac{\|g*v\|_g'}{\|v\|_g'} \le e^{l_2-l_1}. \tag{g}$$

This norm induces a distance on the affine domain B_g^+, which we denote by δ_g'. It does not induce a distance on $\overline{\Omega}$ because the intersection $\mathbb{P}(V_g^<)\cap\overline{\Omega} = \{x_g^-\}$ is not empty, but it none the less induces a distance on Ω. We can then calculate $\phi_b^{\delta_g'}$, but Lemma 14 does not apply.

In the distance δ_g', the calculations are simplified, and for $y_0\in\Omega$

$$\begin{aligned}\phi_\beta^{\delta_g'}(g\cdot y_0) &= \inf\left\{\frac{\delta_g'(g\cdot y_0, y)}{\delta_g'(g\cdot y_0, x)^\beta}\;\middle|\; x,y\in\partial B_\Omega(g\cdot y_0, r_0)\right\}\\ &= \inf\left\{\frac{\delta_g'(g\cdot y_0, g\cdot y)}{\delta_g'(g\cdot y_0, g\cdot x)^\beta}\;\middle|\; x,y\in\partial B_\Omega(g\cdot y_0, r_0)\right\}\\ &= \inf\left\{\frac{\|g*(\phi_g^{-1}(y_0)-\phi_g^{-1}(y))\|_g'}{\|g*(\phi_g^{-1}(y_0)-\phi_g^{-1}(x))\|_g'^\beta}\;\middle|\; x,y\in\partial B_\Omega(g\cdot y_0, r_0)\right\}\end{aligned}$$

because g is an isometry from the boundary $\partial B_\Omega(y_0,r_0)$ to the boundary $\partial B_\Omega(g\cdot y_0,r_0)$. Then, using the inequalities (g),

$$\geq e^{l_{n+1}-l_1}(e^{l_2-l_1})^{-\beta}\inf\left\{\frac{\|\phi_g^{-1}(y^0)-\phi_g^{-1}(y)\|_g'}{\|\phi_g^{-1}(y^0)-\phi_g^{-1}(x)\|_g'^{\beta}}|x,y\in\partial B_\Omega(g\cdot y_0,r_0)\right\}$$
$$\geq e^{(\beta-\beta_g)(l_1-l_2)}\phi_\beta^{\delta_g'}(y_0)$$

and since $(\beta-\beta_g)(l_1-l_2)\geq 0$, we obtain the following inequality

$$\phi_\beta^{\delta_g'}(g\cdot y_0)\geq\phi_\beta^{\delta_g'}(y_0).$$

If y_0 belongs to K, it remains to compare $\phi_\beta^{\delta_g'}(y_0)$ and $\phi_\beta^{\delta_g}(y_0)$ and the same with $\phi_\beta^{\delta_g'}(g\cdot y_0)$ and $\phi_\beta^{\delta_g}(g\cdot y_0)$, if g is (r,ε)-loxodromic. This is what we do in the following lemma.

Lemma 24. *There exists a constant $E>0$ such that for all (r,ε)-loxodromic g and for all y_0 in K,*

$$\frac{1}{E}\phi_\beta^{\delta_g'}(y_0)\leq\phi_\beta^{\delta_g}(y_0)\leq E\phi_\beta^{\delta_g'}(y_0)$$
$$\textit{and }\frac{1}{E}\phi_\beta^{\delta_g'}(g\cdot y_0)\leq\phi_\beta^{\delta_g}(g\cdot y_0)\leq E\phi_\beta^{\delta_g'}(g\cdot y_0)$$

Proof. We define

$$K'=\{x\in\Omega\mid d_\Omega(x,K)\leq r_0\}.$$

The intersection of K' with the boundary $\partial\Omega$ is empty, so there exists a number e such that $\delta_0(K',\partial\Omega)\geq e>0$. If g is (r,ε)-loxodromic, then for the constant C_r from Corollary 19

$$\delta_g(K',\partial\Omega)\geq\frac{e}{C_r}.$$

This means that, if y_0 belongs to K, the ball $B_\Omega(y_0,r_0)$ is far from $\mathbb{P}(V_g^<)$ (the distance being greater than e/C_r), which implies the same property for $B_\Omega(g\cdot y_0,r_0)$ and thus the equivalence of the distances. More precisely, we have the following lemma.

Lemma 25. *Let $\eta>0$. Let g a loxodromic element, and define*

$$B_g^{\eta,\delta_g}=\{x\in\mathbb{P}(V)\mid\delta_g(x,\mathbb{P}(V_g^<))\geq\eta\}.$$

There exists a constant $F>0$ not depending on g for which the distances δ_g and δ_g' are equivalent on the set B_g^{η,δ_g}:

$$\textit{for all } x,y\in B_g^{\eta,\delta_g},\ \frac{1}{F}\delta_g(x,y)\leq\delta_g'(x,y)\leq F\delta_g(x,y).$$

Proof. We denote by $(e_1,\dots,e_{n+1})$ the canonical basis for $\mathbb{R}^{n+1}=W$. The usual Euclidean norm on W defines a distance δ on $\mathbb{P}(W)$.

Define $W^<:=\mathbb{R}e_2\oplus\cdots\oplus\mathbb{R}e_{n+1}$. The restriction of the Euclidean norm to $W^<$ defines a distance δ' on the affine patch $\mathbb{P}(W)\backslash\mathbb{P}(W^<)$.

Let g be a loxodromic element in $\mathrm{PGL}(V)$, and $v_1,\dots,v_{n+1}$ the normalized eigenvectors of g. The linear transformation from V to W that sends v_i to e_i induces a bijection of B_g^{η,δ_g} with

$$B^{\eta,\delta_g}:=\{x\in\mathbb{P}(W)\mid\delta(x,\mathbb{P}(W^<))\geq\eta\}.$$

This bijection is furthermore an isometry of $(B_g^{\eta,\delta_g},\delta_g)$ to $(B^{\eta,\delta},\delta)$, and also an isometry from $(B_g^{\eta,\delta_g},\delta_g')$ to $(B^{\eta,\delta},\delta')$. The constant F in the equivalence of the distances δ and δ' is the same. □

The number η is from now on fixed to e/C_r. For all y_0 in K and all (r, ε)-loxodromic g, the ball $B_\Omega(y_0, r_0)$ is a subset of B_g^{η,δ_g}.

The ball B_g^{η,δ_g} is the image in an affine chart ϕ_g of the ball of $(V_g^<, \|\cdot\|'_g)$ centered at 0 and of radius

$$R = \frac{2-\eta^2}{\sqrt{4\eta^2 - \eta^4}}.$$

Since the action of g on $V_g^<$ is a contraction, the image under g of the ball $B_\Omega(y_0, r_0)$ is a subset of B_g^{η,δ_g}.

Since on the balls $B_\Omega(y_0, r_0)$ and $B_\Omega(g \cdot y_0, r_0)$ the distances δ_g and δ'_g are equivalent up to the constant F from the previous lemma, this implies that, for $E = F^{1+\beta}$,

$$\frac{1}{E}\phi_\beta^{\delta'_g}(y_0) \leq \phi_\beta^{\delta_g}(y_0) \leq E\phi_\beta^{\delta'_g}(y_0)$$

$$\text{and } \frac{1}{E}\phi_\beta^{\delta'_g}(g \cdot y_0) \leq \phi_\beta^{\delta_g}(g \cdot y_0) \leq E\phi_\beta^{\delta'_g}(g \cdot y_0).$$

This concludes Lemma 24. □

We can now conclude: for all x_0 in Ω, there exists y_0 in K and (r, ε)-loxodromic g in Γ such that $x_0 = g \cdot y_0$, so by equations (h) and (i),

$$\begin{aligned}
\phi_\beta^{\delta_0}(x_0) &\geq \frac{1}{D}\phi_\beta^{\delta_g}(x_0) \geq \frac{1}{DE}\phi_\beta^{\delta'_g}(x_0) \\
&\geq \frac{1}{DE}\phi_\beta^{\delta'_g}(y_0) \geq \frac{1}{(DE)^2}\phi_\beta^{\delta_0}(y_0) \\
&\geq \frac{c}{(DE)^2}
\end{aligned}$$

thus, $\psi^{\delta_0}(\beta)$ is strictly positive and the boundary $\partial\Omega$ is β-convex, which finishes the proof of Proposition 23.

□

To be complete, we need to prove the following fact used above.

Proposition 26. *Let Ω be a strictly convex divisible domain divided a by group Γ. If the domain Ω is not an ellipsoid, then there exist numbers $r \geq \varepsilon > 0$ and a compact $K \subset \Omega$ such that:*

for all x_0 in Ω, there exists y_0 in K and g in Γ such that
g is (r, ε)-loxodromic and $x_0 = g \cdot y_0$

Proof. According to Theorem 3.6 of [2], the group Γ is Zariski dense and thus contains a loxodromic element [10]. It is a direct consequence of the theorem of Abels, Margulis and Soĭfer [1] that there exists a finite part F of Γ and $r \geq \varepsilon > 0$ for which we have the property:

for all g in Γ, there exists h in F such that
gh^{-1} is (r, ε)-loxodromic.

Furthermore, by hypothesis there exists a compact K_0 in Ω such that $\Omega = \Gamma \cdot K_0$. The compact $K = F \cdot K_0$ satisfies the conclusions of the proposition. □

A Appendix. Quasi-Isometries.

The ideas used here can be easily adapted to show that the injection of Γ into $\mathrm{PSL}(V)$ is a quasi-isometric embedding.

Definition A.1. Let (X, d_X) and (Y, d_Y) be metric spaces.

A transformation ϕ from X to Y is a *quasi-isometric embedding* if there exist (K, C) such that, for all x, x' in X,

$$\frac{1}{K} d_X(x, x') - C \leq d_Y(\phi(x), \phi(x')) \leq K d_x(x, x') + C.$$

We also say (K, C)-quasi-isometric embedding.

Furthermore, ϕ is a *quasi-isometry* if, for all y in Y, there exists x in X with $d_Y(y, \phi(x)) \leq C$.

If Γ is a finitely generated group, it has a metric d_Γ defined by the length of the words in a generating set S. The quasi-isometry class of this metric does not depend on the generators [7].

Let g in $\mathrm{PSL}_{n+1}(\mathbb{R})$. We denote by $\mu_+ = \mathrm{diag}(\mu_1, \ldots, \mu_{n+1})$, with $\mu_1 \geq \cdots \geq \mu_{n+1}$ its Cartan component, i.e. $g = k \exp(\mu_+) l$ for k and l in $\mathrm{PSO}(n+1, \mathbb{R})$.

The following two lemmas will be useful.

Lemma A.2. *Let $d_{PSL_{n+1}(\mathbb{R})}$ be a left-invariant metric on $PSL_{n+1}(\mathbb{R})$ and $\|\cdot\|$ a norm on the diagonal matrices with trace zero coming from the Riemannian metric.*

There then exist (K, C) such that, for all g, and μ_+ the Cartan component of g:

$$\frac{1}{K} \|\mu_+\| - C \leq d_{PSL_{n+1}(\mathbb{R})}(g, id) \leq K \|\mu_+\| + C.$$

Proof. By the invariance and compactness of $PSO_{n+1}(\mathbb{R})$, it suffices to show the inequalities for the subgroup of diagonal matrices. This comes from the equivalence of norms on a finite-dimensional vector space. □

Lemma A.3. *For all $r \geq \varepsilon > 0$, there exists a constant c such that, for all (r, ε)-proximal g in $PSL_{n+1}(\mathbb{R})$, l_1 the logarithm of the modulus of the biggest eigenvalue of g, μ_1 the largest eigenvalue of the Cartan component of g, we have:*

$$c\mu_1 \leq l_1 \leq \mu_1.$$

Proof. This is Lemma 1.3 of [3] □

Proposition A.4. *Let $\Omega \subset \mathbb{P}^n(\mathbb{R})$ be a strictly convex divisible domain and Γ a subgroup of $PSL_{n+1}(\mathbb{R})$ dividing Ω.*

The injection of Γ into $PSL_{n+1}(\mathbb{R})$ is then a quasi-isometric embedding.

Proof. In fact, only the left inequality in Definition A1 is not immediate.

Since the group Γ acts properly on Ω with a compact quotient, the orbit transformation

$$\begin{aligned} (\Gamma, d_\Gamma) &\longrightarrow (\Omega, d_\Omega) \\ g &\longmapsto g \cdot x \end{aligned}$$

is a quasi-isometry (x is some point in Ω). There then exists a constant K such that, for all g in Γ,

$$d_\Omega(g \cdot x, x) \geq \frac{1}{K} l_\Gamma(g).$$

(l_Γ is the length of the words in Γ).

According the the theorem of Abels, Margulis and Soĭfer, there exists $r \geq \varepsilon > 0$ and a finite part F of Γ such that for all g in Γ, there exists h in F such that gh^{-1} is (r, ε)-proximal.

Furthermore, it is easy to see that there exists $R > 0$ such that, for all g in Γ, if g^{-1} is (r, ε)-proximal, x_g^+ and x_g^- are fixed attracting and repelling points of g in $\mathbb{P}^n(\mathbb{R})$, then there exists y belonging to the intersection of Ω and the line (x_g^+, x_g^-) such that $d_\Omega(y, x) \leq R$. Letting $l_1 \geq \cdots \geq l_{n+1}$ be the logarithms of the norms of the eigenvalues of g, we have

$$d_\Omega(g \cdot y, y) = l_1 - l_{n+1},$$

where

$$l_1 - l_{n+1} \geq d_\Omega(g \cdot x, x) - 2R \geq \frac{1}{K} l_\Gamma(g) - 2R.$$

Furthermore, Lemma A3 applies to g^{-1} and allows us to show that

$$\|\mu_+\| := |\mu_1 - \mu_{n+1}| \geq \frac{1}{c}(l_1 - l_{n+1}).$$

Let (K, C) be the constants from Lemma A2. For $g \in \Gamma$, there exists an h in F such that $g^{-1}h$ is (r, ϵ)-proximal, so

$$K d_{\mathrm{PSL}_{n+1}(\mathbb{R})}(h^{-1}g, \mathrm{id}) + C \geq \frac{1}{c}\left(\frac{1}{K} l_\Gamma(h^{-1}g) - 2R\right),$$

and we get:

$$l_\Gamma(h^{-1}g) \geq l_\Gamma(g) - \max_{h \in F} l_\Gamma(h)$$
$$d_{\mathrm{PSL}_{n+1}(\mathbb{R})}(g, \mathrm{id}) \geq d_{\mathrm{PSL}_{n+1}(\mathbb{R})}(h^{-1}g, \mathrm{id}) - \max_{h \in F} d_{\mathrm{PSL}_{n+1}(\mathbb{R})}(h, \mathrm{id})$$

obtaining the desired conclusion. □

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