ABSTRACT

Title of Thesis: PROJECTIVE DEFORMATIONS OF TRIANGLE TILINGS

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A hyperbolic triangle group is the group generated by reflections in the sides of a triangle in hyperbolic space. For a given hyperbolic triangle group, we find a one-parameter group of representations into GL(3,$\mathbb{R}$) and associated invariant cones. We show that the representations are faithful and that the cones are sharp. We then apply the results of Guichard to approximate the Hölder continuity of the boundaries of the cones. We conjecture that this may be directly calculated by considering only the Coxeter elements of the triangle group.

PROJECTIVE DEFORMATIONS OF TRIANGLE TILINGS

by

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1 Background

1.1 Cones

A *cone* U in $\mathbb{R}^n$ is a domain invariant under positive homotheties:

$$\lambda u \in U \text{ for any } u \in U, \lambda > 0.$$

A cone is *sharp* (or *properly convex*, or, in French, *saillant*) if its closure $\overline{U}$ does not contain a full line. It is *strictly convex* if any codimension-1 subspace of $\mathbb{R}^n$ intersects the boundary ∂U of U in at most a ray. The *dual* U^* of a cone U is the set of linear functionals on $\mathbb{R}^n$ that are strictly positive on U.

A space X with an action of a group G is called *homogeneous* if G acts transitively on X:

$$\text{for some } x \in X, Gx = X.$$

Relaxing this condition, call X *quasi-homogeneous* if there exists a compact K and a group G acting on X such that $GK = X$. If the quotient X/G is furthermore Hausdorff, X is called *divisible*, and G is said to *divide* X. If G is acting properly ($gK \cap K \neq \emptyset$ for only finitely many g) and is torsion-free, then X/G is a manifold. If G acts properly and is virtually torsion-free (has a subgroup of finite index that is torsion-free), then X/G is called a (good) *orbifold*.

The following invariant will allow us to study the boundaries of divisible domains:

Definition 1.1. A transformation g in $\mathrm{GL}(n, \mathbb{R})$ is called *hyperbolic* (or *loxodromic*) if its eigenvalues have distinct norms. For such a transformation, let $l_1 > \ldots > l_n$ be the logarithms of the norms of the eigenvalues of g. Define

$$\alpha_g := \frac{l_1 - l_n}{l_1 - l_{n-1}},$$
$$\beta_g := \frac{l_1 - l_n}{l_1 - l_2}.$$

Definition 1.2. For $G \subset \mathrm{GL}(n, \mathbb{R})$ define:

$$\alpha_G := \inf_{g \in G,\, g \text{ hyperbolic}} \alpha_g, \quad \beta_G := \sup_{g \in G,\, g \text{ hyperbolic}} \beta_g$$

It is easy to show that $\alpha_{g^{-1}}^{-1} + \beta_g^{-1} = 1$, and the same equality holds for groups: $\alpha_G^{-1} + \beta_G^{-1} = 1$.

Note that for $n = 3$, the definitions of α_g and β_g coincide:

$$\alpha_g = \beta_g = \frac{l_1 - l_3}{l_1 - l_2}.$$

1.2 Projective Geometry

The *real projective space* $\mathbb{RP}^n$ is the space of lines in $\mathbb{R}^{n+1}$. Since cones are invariant under scaling it is the natural setting for drawing them and analyzing their properties. For $p \in \mathbb{RP}^n$, a nonzero vector in the line $p \subset \mathbb{R}^{n+1}$ gives the *homogeneous coordinates* for p. These are defined up to non-zero scaling. The projection map $\mathbb{R}^{n+1}\backslash\{0\} \to \mathbb{RP}^n$ induces an action of $\mathrm{GL}(n+1,\mathbb{R})$ on $\mathbb{RP}^n$. The image of $\mathrm{GL}(n+1,\mathbb{R})$ under this homomorphism into $\mathrm{Aut}(\mathbb{RP}^n)$ is called $\mathrm{PGL}(n+1,\mathbb{R})$ and is isomorphic to $\mathrm{SL}(n+1, \mathbb{R})$. We will use this isomorphism to think of $\mathrm{PGL}(n+1,\mathbb{R})$ as a subgroup of $\mathrm{GL}(n+1,\mathbb{R})$. The action of $\mathrm{PGL}(n+1,\mathbb{R})$ is $(n+1)$-transitive on points in general position (no proper subset of the $n+1$ points is linearly dependent).

$\mathbb{RP}^n$ is a manifold, with the standard coordinate patch given by dividing out the last coordinate:

$$\begin{bmatrix} x_1 \\ \vdots \\ x_{n+1} \end{bmatrix} \mapsto \begin{bmatrix} x_1/x_{n+1} \\ \vdots \\ x_n/x_{n+1} \end{bmatrix}.$$

Other coordinate patches can be found by dividing out by other coordinates, or by using the transitivity of the $\mathrm{PGL}(n+1,\mathbb{R})$ action.

1.3 Hyperbolic Geometry

Definition 1.3. Let $B\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}, \begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = x^2 + y^2 - z^2$ be the Lorentz inner product on $\mathbb{R}^3$. Denote the corresponding norm by $\|\cdot\|_B$ and consider the upper component of the hyperboloid of vectors with norm squared -1:

$$\mathbb{H}_h = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; \left\| \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right\|_B^2 = -1 \text{ and } z > 0 \right\}$$

By restriction, B induces a positive definite Riemannian metric on $\mathbb{H}_h$. $\mathbb{H}_h$ with this metric is known as the *hyperboloid model* of the *hyperbolic plane* $\mathbb{H}$. The projectivization of $\mathbb{H}_h$ into the standard coordinate patch of $\mathbb{RP}^2$ is known as the *Klein unit disk* $\mathbb{H}_K$.

Definition 1.4. The *complex projective line* $\mathbb{CP}^1$ is the set of equivalence classes of vectors in $\mathbb{C}^2\backslash\{0\}$ up to scaling, with the induced automorphism group $\mathrm{PGL}(2,\mathbb{C})$. As with $\mathbb{RP}^n$, the standard coordinate patch is given by dividing out the last coordinate:

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \frac{z_1}{z_2}.$$

We thus get an identification of $\mathbb{CP}^1$ with $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$, the Riemann sphere, on which GL(2, $\mathbb{C}$) acts by linear fractional transformations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} : z \mapsto \frac{az+b}{cz+d}.$$

Definition 1.5. Let $\mathbb{H}_u = \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\} \subset \mathbb{CP}^1$ be the upper half-plane, with the Riemannian metric $ds^2 = \frac{dx^2+dy^2}{y^2}$. This is the *Poincaré upper half-plane* model of the hyperbolic plane.

The upper half plane can be transformed to a unit disk using an element of PGL(2, $\mathbb{C}$). The transformation induces a Riemannian metric on the unit disk, and we define the disk with this metric to be the *Poincaré unit disk model* $\mathbb{H}_D$ of hyperbolic space.

These spaces are all isomorphic Riemannian manifolds.

The action GL(2, $\mathbb{C}$) on the Riemann sphere restricts to the action of SL(2, $\mathbb{R}$) on the upper half-plane and gives all the orientation-preserving automorphisms of $\mathbb{H}_u$. These are classified by trace:

Lemma 1.6. *Let $M \in$ SL(2, $\mathbb{R}$) be a an automorphism of $\mathbb{H}_u$. Then M is, up to multiplication by $-I$, conjugate to one of the following, identified by trace:*

- $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} : z \mapsto z$, *the identity transformation, with trace 2*
- $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} : z \mapsto z+1$, *a parabolic transformation, with trace 2*
- $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} : z \mapsto \lambda^2 z$, $(\lambda \neq 1)$, *a hyperbolic transformation, with trace* $\lambda + \lambda^{-1} > 2$
- $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, *an elliptic transformation, with trace* $2\cos\theta < 2$

Note that an elliptic transformation is GL(2, $\mathbb{C}$)-conjugate to $\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} :$ $z \mapsto e^{2i\theta}$, *a rotation fixing the origin and* ∞.

Proof. If M has real eigenvalues, it can be diagonalized over $\mathbb{R}$ to be in one of the first three forms. In the last case, the eigenvalues are distinct and we get the fourth form by first diagonalizing over $\mathbb{C}$, and then converting back to the real matrix. $\square$

For further information on hyperbolic geometry, see [9].

1.4 Coxeter Groups

A *Coxeter group* W on n generators has the presentation

$$W = \left\langle s_i \;\middle|\; (s_i s_j)^{n_{ij}} = s_i^2 = 1 \right\rangle, \; n_{ij} \in \mathbb{N}, \; i, j = 1, \dots, n.$$

A Coxeter group with a choice of generators $S = \{s_i\}$ is called a *Coxeter system.*

Coxeter groups with two generators are the dihedral groups, and ones with 3 generators are called *triangle groups.* In the latter case, we denote the three numbers n_{ij} by p, q, r, and call a triangle group a:

- Spherical Triangle Group if $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$
- Euclidean Triangle Group if $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$
- Hyperbolic Triangle Group if $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$

Equivalently, a (p, q, r) triangle group can be defined as the group generated by reflections in the sides of a triangle with interior angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$ (see [7]). The above classification reflects the availability of such triangles in the corresponding geometry.

Given an element w of W, we may write it as words in s_i in many ways. Define $l(w)$ to be the length of the shortest word in s_i that corresponds to w. This extends to the word metric $d(w, w') := l(w^{-1}w')$ on W. The word metric on Coxeter groups has the following essential property:

$$l(s_i w) = l(w) + 1 \text{ or } l(w) - 1.$$

For a non-trivial word w, we may always find an s_i such that $s_i w$ has length $l(w) - 1$, which allows inductive arguments.

Given $S' = \{r_1, \dots, r_k\} \subset \{s_1, \dots, s_n\}$, define

$$W_{S'} := \langle r_1, \dots, r_k \rangle \subset W$$

Note that the word metric on $W_{S'}$ does not necessarily agree with that on all of W (an element $w \in W_{S'}$ may have a shorter representation in W).

There are two important classes of elements of W:

- An element w is called *essential* if it is not conjugate to an element of $W_{S'}$ for any proper subset S' of the generators.
- An element of the form $\tau(s_1) \cdots \tau(s_n)$, where τ is any permutation of the generators, is called a *Coxeter element.* Up to conjugacy there are only two Coxeter elements in a triangle group: $s_1 s_2 s_3$ and its inverse $s_3 s_2 s_1$.

We will prove the following fact in Section 4.1:

Theorem 1.7. *Coxeter elements of hyperbolic triangle groups have infinite order.*

Corollary 1.8. *Coxeter elements of hyperbolic triangle groups are essential.*

Proof. Any proper subgroup $W_{S'}$ of a triangle group is of finite order. $\square$

1.5 Hilbert Metric

Definition 1.9. Let Ω be a convex domain, and $x, y \in \Omega$. The Hilbert distance between x and y is

$$d_\Omega(x, y) := \frac{1}{2} \inf \log\, [a, x; y, b],$$

where the infimum is taken over all a and b in Ω such that $\overline{xy} \subseteq \overline{ab}$ and $[a, x; y, b]$ is the cross-ratio

$$[a, x; y, b] := \frac{|x - b|\,|y - a|}{|x - a|\,|y - b|}.$$

Note that the infimum is attained when $a, b \in \partial\Omega$. Due to the projective invariance of the cross-ratio, the Hilbert metric is invariant under the projective automorphisms of Ω. For a proof of the triangle inequality, see [11].

Straight lines in $\mathbb{RP}^n \cap \Omega$ are geodesics in the Hilbert metric. For strictly convex domains Ω, these are the only geodesics, and the Hilbert metric induces a norm on the tangent spaces of Ω.

Example 1. *Let Ω be the unit ball in $\mathbb{R}^n$. Then $d_H = d_K$, the metric on the Klein model of hyperbolic space.*

Proof. Let $x, y \in \Omega$. Since Ω is homogeneous with respect to its projective automorphisms, we may assume y is the origin, and furthermore rotate such that x lies on the x-axis, with x-coordinate $0 < a < 1$. Thus, we are reduced to the case $n = 1$.

$$d_H(0, a) = \frac{1}{2} \log\, [-1, 0; a, 1] = \frac{1}{2} \log\, \frac{a + 1}{a - 1}.$$

To calculate $d_K(0, x)$, we switch to the hyperbola model of $\mathbb{H}^1 \subset \mathbb{R}^{1,1}$, defined by the equation $x^2 - y^2 = -1$ (recall that the metric on hyperbolic space is the restriction of the Lorentz inner product to this hyperbola). Under the equivalence between the Klein and hyperbolic models (Figure 1), we associate:

$$0 \Leftrightarrow (0, 1), \quad a \Leftrightarrow (b, b/a), \text{where } b = \frac{a}{\sqrt{1 - a}}.$$

To find the distance, we integrate along the hyperbola $y = \sqrt{x^2 + 1}$ in the $\mathbb{R}^{1,1}$ metric $\sqrt{dx^2 - dy^2}$:

$$d_K(0, a) = \int_0^b \sqrt{1 - \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^b (x^2 + 1)^{-1/2} dx$$

Using the substitution $x = \sinh(u)$, this reduces to $d_K(0, a) = \sinh^{-1} b$. Plugging in b,

$$d_K(0, a) = \frac{1}{2} \log \frac{1 + a}{1 - a} = d_H(0, a).$$

$\square$

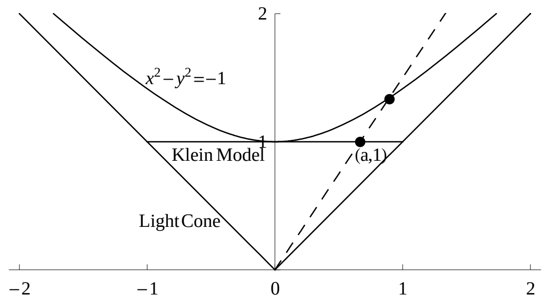

Figure 1: The Klein model of hyperbolic 1-space can be viewed as the open segment between $(-1, 1)$ and $(1, 1)$. The hyperbola model is given by the hyperbola $x^2 - y^2 = -1$. Given a point $(a, 1)$ in the Klein model, we identify it with a point in the hyperbola using a straight line from the origin (dashed line).

2 Convex Cones

2.1 Tits Cone

We now define the Tits cone and discuss its basic properties, basing proofs on [3, 4, 10].

For this section, fix a Coxeter system (W, S) with $S = \{s_1, \ldots s_n\}$ and a basis $\{e_s\}_{s \in S}$ for $V = \mathbb{R}^n$. Let $B(\cdot, \cdot)$ be the bilinear form on V defined by:

$$B(e_{s_i}, e_{s_j}) = \begin{cases} 1 & i = j \\ -\cos \frac{\pi}{n_{ij}} & i \neq j \end{cases}.$$

Definition 2.1. (Tits cone) For each $s \in S$, define:

$$\begin{aligned} R_s(x) &:= x - 2B(x, e_s)e_s, \\ H_s &:= \{x \in V \mid B(x, e_s) = 0\}, \\ A_s &:= \{x \in V \mid B(x, e_s) > 0\}, \\ C &:= \bigcap_{s \in S} A_s. \end{aligned}$$

Each R_s is a reflection over H_s (see Figure 2). Define the representation ρ of W into GL(3, $\mathbb{R}$) by

$$\rho(s_i) := R_{s_i}$$

and set $G := \rho(W)$. The *Tits cone* is defined as the image of $\overline{C}$ under the action of W:

$$U := \bigcup_{g \in G} g\overline{C}.$$

As we show below, W acts freely on the images of $\overline{C}$, which is a fundamental domain for the action. The proof stems from the following lemma for the dihedral case:

Lemma 2.2. *Let* $W = \langle a, b \mid (ab)^n = a^2 = b^2 = 1 \rangle$*, and* $\rho : W \to O(2, \mathbb{R})$ *such that, in polar coordinates:*

- $\rho(a)$ *is the reflection over the line* $\theta = 0$*,*
- $\rho(b)$ *is the reflection over* $\theta = \frac{\pi}{n}$*.*

Let C *be the region* $\{(r, \theta) \mid r > 0,\ \ 0 < \theta < \frac{\pi}{n})\}$*. Then, for* $w \in W$*,*

$$\rho(w)(C) \subset A_a = \{(x, y) \mid x > 0\} \text{ if and only if } l(aw) < l(w).$$

Proof. We assign each image C' of the fundamental domain a word w of minimal length in $\{a, b\}$ such that $\rho(w)(C) = C'$. C is assigned the null word, its neighbors are assigned the two words of length 1: a, b. Their neighbors are images of C under the words ab and ba. To see that those are minimal, note that all words of smaller length are already taken. Proceeding by induction, a word representing C' starts with a if and only if it is not in A_a (Figure 2). $\square$

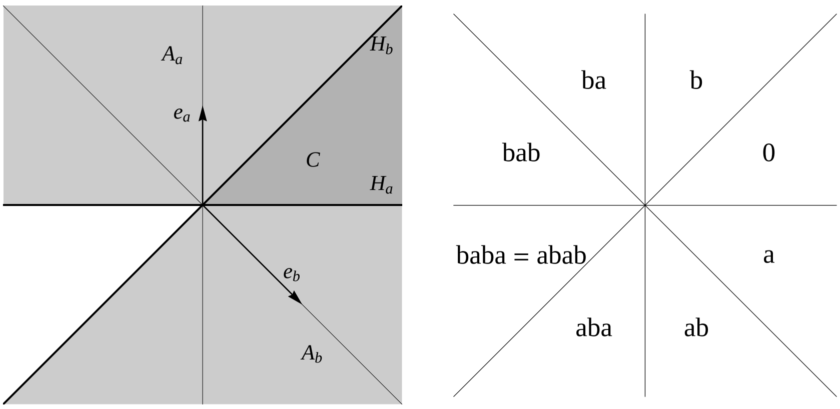

Figure 2: Notation and Lemma 2.2 for D_4, in a basis orthogonal with respect to B.

Lemma 2.3. *The corners of the Tits cone are dihedral: for any distinct* i, j *there is a basis such that* R_{s_i}*,* R_{s_j}*, and* $A_i \cap A_j$ *satisfy the conditions in Lemma 2.2.*

Proof. Let $n = n_{ij} = |s_i s_j|$. Restricted to the subspace $\langle e_i, e_j \rangle$, the reflections are represented by matrices

$$R_i = \begin{bmatrix} -1 & 2\cos\frac{\pi}{n} \\ 0 & 1 \end{bmatrix}, \quad R_j = \begin{bmatrix} 1 & 0 \\ 2\cos\frac{\pi}{n} & -1 \end{bmatrix}.$$

The product

$$R_iR_j = \begin{bmatrix} -1 + 4\cos^{\frac{2\pi}{n}} & -2\cos\frac{\pi}{n} \\ -2\cos\frac{\pi}{n} & -1 \end{bmatrix}$$

has trace $2\cos\frac{2\pi}{n}$ and determinant 1. The two eigenvalues are then inverses of each other, and $\lambda + \lambda^{-1} = 2\cos\frac{2\pi}{n} < 2$. Since this expression is greater than two for λ real, λ must be imaginary. Thus, under the right basis R_1R_2 is a rotation of order n.

Now, extend e_i to an orthonormal basis with respect to B. In this basis, R_i is reflection over the line $L_i - \{\theta = 0\}$, as desired. In the same basis, R_j must be an orthogonal reflection such that R_iR_j is of order n, so it must be reflection over a line $L_j = \{\theta = \pm\frac{k\pi}{n}\}$ for some integer k.

Now, the angle ϕ between e_i and e_j is given by B:

$$B(e_i, e_j) = \|e_i\|_B \|e_j\|_B \cos\phi$$

so that

$$-\cos\frac{\pi}{n} = \cos\phi.$$

This forces $\phi = \pm\left(\pi - \frac{\pi}{n}\right) \pmod{2\pi}$, and since by construction $B(e_i, L_i) = 0, B(e_j, L_j) = 0$,

$$\theta = \pm\frac{\pi}{n}.$$

Conjugating by R_i if necessary, we may assume this sign is positive. Thus, up to an order-2 rotation $A_i \cap A_j$ must be $\left\{(r,\theta) \;\middle|\; r > 0,\ \ 0 < \theta < \frac{\pi}{n}\right\}$. $\square$

Theorem 2.4. (Tits) *We use the above notation for the Tits cone. Then*

(P) For each $w \in W$ and $s \in S$, either
$wC \subset A_s$ and $l(w) = l(sw) - 1$, or
$wC \subset sA_s$ and $l(w) = l(sw) + 1$.

Informally, if wC is on the "negative" side of the cone with respect to s, then there is a minimal word equivalent to w that starts with s.

Proof. We split the assertion (P) into smaller statements for an inductive argument:

(P_n) Let $w \in W, l(w) = n$. For each $s \in S$, either
$wC \subset A_s$ and $l(w) = l(sw) - 1$, or
$wC \subset sA_s$ and $l(w) = l(sw) + 1$.

To deal with cases where w doesn't start with s (for any minimal word equivalent to w), we need the following assertions:

(Q_n) Let $w \in W, l(w) = n$. For each $s, t \in S$, there exists $u \in W_{s,t}$ such that
$wC \subset uC$ and $l(w) = l(u) + l(u^{-1}w)$.

The intuitive idea is the following: taking $s, t \in S$, split the Tits cone into sectors given by the action of the subgroup $W_{s,t}$. If wC is in a sector indexed by a word $u \in W_{s,t}$, then w starts with u (there is an equivalent minimal-length word that starts with u).

P_0 and Q_0 are trivial. We now prove that P_n and Q_n imply P_{n+1} and that P_{n+1} and Q_n imply Q_{n+1}.

$\underline{\mathrm{P}_n \text{ and } \mathrm{Q}_n \text{ imply } \mathrm{P}_{n+1}.}$ Let $w \in W$ with $l(w) = n + 1$. If $l(sw) = l(w) - 1$ (intuitively, w starts with s), then apply P_n to sw and then multiply by s to prove P_{n+1}.

If $l(sw) > l(w)$, pick some t such that $l(tw) < l(w)$. Set $w' = tw$. Applying Q_n to w', we get $u \in W_{s,t}$ such that $w'C \subset uC$ and $l(w') = l(u) - l(u^{-1}w')$. So $wC = tw'C \subset tuC$. We now analyze $v = tu$. By Lemmas 2.3 and 2.2, $W_{u,v}$ has property P. Thus, one of the following is true:

1. $v(A_s \cap A_t) \subset A_s$, which implies $wC \subset vC \subset A_s$, or
2. $v(A_s \cap A_t) \subset sA_s$, which would show that $wC \subset sA_s$.

Assume by way of contradiction that the latter is true. Then it is also true that $l'(sv) < l'(v)$ (where l' denotes the word metric in the group $W_{s,t}$). We calculate:

$$\begin{aligned} l(sw) = l(stw') = l((stu)(u^{-1}w')) &\leq l'(stu) + l(u^{-1}w') \\ < l'(tu) + l(u^{-1}w') &= l'(tu) - l'(u) + l(w') \leq l(w). \end{aligned}$$

That contradicts the assumption that $l(sw) > l(w)$.

$\underline{\mathrm{P}_{n+1} \text{ and } \mathrm{Q}_n \text{ imply } \mathrm{Q}_{n+1}.}$ Let $s, t \in S, w \in W$ with $l(w) = n+1$. If $wC \subset A_s \cap A_t$ then we are done with $u = 1$. Otherwise, choose s to be the letter such that $wC \not\subseteq A_s$. By P_{n+1}, $l(sw) = l(w) - 1 = n$ and now Q_n applies to sw, giving us $v \in W_{s,t}$ such that $swC \subset v(A_s \cap A_t)$ and $l(sw) = l(v) + l(v^{-1}sw)$. We calculate:

$$l(w) = 1 + l(sw) = 1 + l(v) + l(v^{-1}) \geq l(sv) + l((sv)^{-1}w) \geq l(w)$$

So that the last inequality must be an equality, and Q_{n+1} is proven using $u = sv$. □

To summarize, the first assertion is proven by either showing that wC is in the s-negative side sA_s, or by using another generator to reduce the length of the word and use the inductive hypothesis to show that wC must already be in A_s. The second assertion is proven by making one move in the rotation that brings wC to the sector $A_s \cap A_t$ and letting induction take care of the remaining rotation.

Corollary 2.5. *Let $w \in W$. Then $wC \cap C \neq \emptyset$ if and only if $w = 1$. Thus, the representation of the Coxeter group in $GL(V)$ is faithful.*

Proof. If $w \neq 1$ then there is some $s \in S$ such that $l(sw) = l(w) - 1$. But then by property (P), $wC \subset sA_s$, so $wC \cap C = \emptyset$, a contradiction. □

Corollary 2.6. *The Tits cone U is convex. That is, if $x, y \in U$, then the segment $\overline{xy} \subset U$.*

Proof. We may assume without loss of generality that $x \in \overline{C}$, the fundamental domain. Pick a $w \in W$ such that $y \in w\overline{C}$. $\overline{C}$ is convex since it is the intersection of the half-planes $\overline{A_s}$. Thus, if $w = 1$, then we are done.

Say $l(w) = n$ and assume inductively that the proposition holds for all shorter words. Consider the intersection $\overline{xz} = \overline{xy} \cap \overline{C}$. z is in the boundary of $\overline{C}$, say in H_s for some $s \in S$. But then $z \in s\overline{C}$, and the minimal word connecting z and y has length $l(sw) = l(w) - 1$ by property P. By induction, the segment $\overline{zy} \subset U$, so $\overline{xy} \subset U$. □

Considering the restrictions placed on U by the existence of the bilinear form provide

Theorem 2.7. (Vinberg [14, 11])*If W is a hyperbolic triangle group, then the closure of the Tits cone contains no line.*

The Tits cone is the domain on which the bilinear form B is positive definite. By strict convexity, this cannot be all of $\mathbb{R}^3$. B must therefore be of signature (2,1). We thus have:

Corollary 2.8. *The projectivization of the Tits cone for a hyperbolic triangle group is projectively equivalent to the Klein unit disk.*

2.2 Kac-Vinberg Cones

The Tits cone construction was modified by Kac and Vinberg in [8] to produce an inhomogeneous cone divided by a triangle group.

Let e_1, e_2, e_3 be the standard basis of $\mathbb{R}^3$. Define $B(\cdot,\cdot)$ to be a bilinear form, with $B_{ij} = B(e_i, e_j)$, such that:

1. $\det[B_{ij}] < 0$.
2. $B_{ii} = 2$, and the other entries are negative integers.
3. $B_{ij}B_{ji} < 4$ for $i \neq j$.
4. $B_{12}B_{23}B_{31} \neq B_{21}B_{32}B_{13}$.

Example 2. $[B_{ij}] = \begin{bmatrix} 2 & -1 & -3 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

The rows of $[B_{ij}]^{-1}$ give vectors l_i such that $B(l_i, e_j) = \delta_{i,j}$. As with the Tits cone, define

$$R_i(v) = v - B(v, e_i)e_i$$

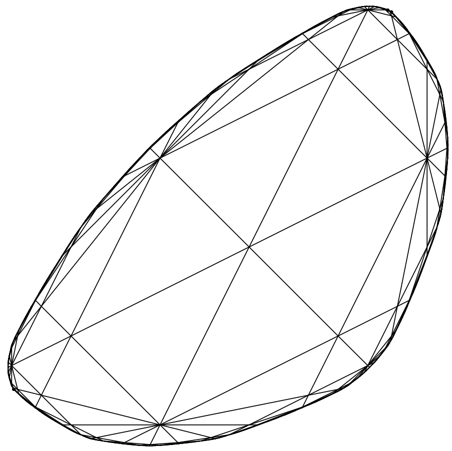

Figure 3: The Kac-Vinberg cone for $[B_{ij}]$ in Example 2.

$$C = \mathbb{R}^+ l_1 \oplus \mathbb{R}^+ l_2 \oplus \mathbb{R}^+ l_3$$
$$G = \langle R_1, R_2, R_3 \rangle$$
$$U = G\overline{C}$$

We refer to U as a Kac-Vinberg cone.

Theorem 2.9. (Theorem 1 in [8]) *Let $U' \subset \mathbb{R}^3$ be a quasi-homogeneous sharp cone whose boundary is twice differentiable except at finitely many points. Then U' is homogeneous, and thus either a triangular cone or an elliptical cone.*

Theorem 2.10. (Theorem 2 in [8]) *A Kac-Vinberg cone U is not homogeneous.*

Proof. Assume by way of contradiction that U is homogeneous. It can be verified directly that the order of R_iR_j is n_{ij} is given by

$$4\cos^2\frac{\pi}{n_{ij}} = B_{ij}B_{ji}.$$

If U were a triangular cone, its rotational symmetries R_iR_j would have to be of order 3. This would force $B_{ij} = -1$ and give a singular matrix.

If U were an elliptical cone, there would be an inner product $(\cdot,\cdot)$ invariant under the symmetries of U. It can then be shown that $2(e_i, e_j) = B_{ji}(e_i, e_i)$. Since the inner product is non-degenerate, and $B_{ij} < 0$, this implies $(e_i, e_j) \neq 0$ for all i, j. The symmetry of the inner product then gives $B_{12}B_{23}B_{31} = B_{21}B_{32}B_{13}$, a contradiction.

Thus, U is not homogeneous, and is not twice differentiable at infinitely many points. $\square$

2.3 Hölder Regularity

While the boundary of a Kac-Vinberg cone is not twice differentiable, it is more than just once differentiable. One can extend the notion of n^{th} derivative to non-integer values of n:

Definition 2.11. Let $\alpha \in (0,1]$ and $f : \mathbb{R} \to \mathbb{R}$.

$$f^{(\alpha)}(x) = \lim_{y \to x} \frac{f(x) - f(y)}{|x-y|^\alpha}.$$

A weaker notion of derivative is more useful than the existence of this infinitesimal quantity. The following definition requires the difference quotient to stay bounded rather than to converge.

Definition 2.12. Let $\alpha \in (0,1]$ and $f : \mathbb{R} \to \mathbb{R}$. f is α-Hölder if there exists a C such that for all $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \leq C\,|x-y|^\alpha.$$

For $\alpha - n \in (0,1]$, f is α-Hölder if the n^{th} derivative $f^{(n)}$ is $(\alpha - n)$-Hölder.

We define C_α to be smallest constant C fulfilling the above inequality, with $C_\alpha = \infty$ if no such C exists.

Example 3. *$f(x) = |x|^\alpha$ is α-Hölder for $\alpha > 0$.*

Proof. In all cases, we are reduced to the case $\alpha \in (0,1]$. Fix $y \in \mathbb{R}$ and define

$$C(y) = \sup_{x \in \mathbb{R}} \frac{|f(x) - f(y)|}{|x-y|^\alpha} = \sup_{x \in \mathbb{R}} \frac{||x|^\alpha - |y|^\alpha|}{|x-y|^\alpha}.$$

The fraction is not defined for $x = y$. However, we fill it in with the value 0 by using l'Hôspital's rule. Since it increases away from y and approaches 1 as x goes to $\pm\infty$, $C(y) = 1$. Since this works for any y, f is α-Hölder with constant $C_\alpha = 1$. □

Definition 2.13. A curve M in $\mathbb{R}^2$ is α-Hölder if it is locally the graph of an α-Hölder function. For $\alpha \in (1,2]$ this is equivalent (Lemma 4.2 of [1]) to the condition that for all compact $K \subset M$ there exists C_K such that for all $p, q \in K$

$$d(q, T_pM) \leq C_K d(p,q)^\alpha,$$

where $T_pM \subset \mathbb{R}^2$ is the tangent line to M at the point p. We use this description of Hölder continuity to define it when M is a submanifold of an arbitrary space endowed with a distance d.

Thus, while Hölder continuity states that the difference quotient $\frac{|f(x)-f(y)|}{|x-y|^\alpha}$ is bounded on the entire domain of f, for locally Hölder functions we require this condition only as x approaches y. For example, the function $f(x) = x^2$ is locally, but not globally, 1-Hölder since the derivative is locally, but not globally, bounded.

Lemma 2.14. (Projective Invariance) *Let $M \subset E \subset \mathbb{RP}^n$ be a curve in an affine patch E of projective space, and f a nonsingular projective transformation such that $f(M) \subset E$. If M is α-Hölder in the metric d given by E, then so is $f(M)$.*

Proof. We only have to show the invariance of α-Hölder continuity on compact subsets of M. Let K be a compact subset of M, and $\widehat{K}$ a compact neighborhood of K in $E \cap f^{-1}(E)$.

f is bi-Lipschitz on $\widehat{K}$: there is a constant c such that for all $x, y \in K$,

$$\frac{1}{c}d(x,y) \leq d(fx, fy) \leq cd(x,y), \text{ given by}$$

$$c = \max\left\{\sup_{x,y\in K} \frac{d(x,y)}{d(fx,fy)}, \sup_{x,y\in K} \frac{d(fx,fy)}{d(x,y)}\right\}.$$

The ratios are defined everywhere since f is differentiable and invertible, and the suprema are attained since $\widehat{K}$ is compact.

Since M is α-Hölder, we have a constant C such that for all $x, y \in K$,

$$d(x, T_y M) \leq Cd(x,y)^\alpha.$$

Extending this inequality using the fact that f is bi-Lipschitz on $\widehat{K}$,

$$\frac{1}{c}d(fx, f(T_y M)) \leq d(x, T_y M) \leq Cd(x,y)^\alpha \leq C(cd(fx,fy))^\alpha.$$

Since f is a projective transformation, $f(T_y M) = T_{fy} f(M)$, and this reduces to

$$\frac{1}{c}d(fx, T_{fy}f(M)) \leq C(cd(fx,fy))^\alpha$$
$$d(fx, T_{fy}f(M)) \leq (Cc^{\alpha+1})d(fx,fy)^\alpha$$

So the compact subset $K \subset f(M)$ is α-Hölder for any compact K. Thus, $f(M)$ is α-Hölder. $\square$

Definition 2.15. Let M be a submanifold of a metric space. $\alpha_M := \sup\{\alpha \leq 2 : M \text{ is } \alpha\text{-Hölder}\}$

Lemma 2.16. *Let $M \subset E \subset \mathbb{RP}^n$ be a submanifold of $\mathbb{RP}^n$ contained in a coordinate patch E. Assume furthermore that M is invariant under a hyperbolic transformation g, with eigenvalues $|\lambda_1| > \ldots > |\lambda_{n+1}|$ and $l_i = log(|\lambda_i|)$. Then, using the Euclidean distance given by E,*

$$\alpha_M \leq \alpha_g = \frac{l_1 - l_{n+1}}{l_1 - l_n}.$$

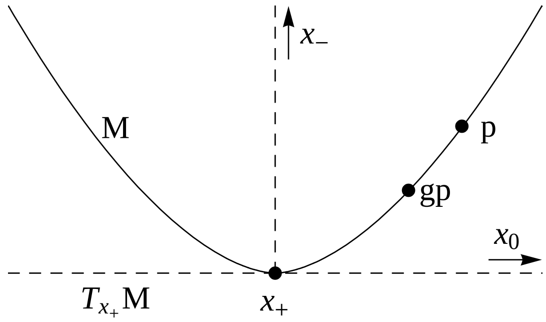

Figure 4: A g-invariant curve $M \subset \mathbb{RP}^2$ viewed in a coordinate patch given by g. The points x_+, x_0, x_- are, respectively, the attracting, saddle, and repelling fixed points of g.

Proof. We consider the case $n = 2$. The general case is essentially identical.

Let $x_+, x_o, x_- \in \mathbb{RP}^2$ be, respectively, the attracting, saddle, and repelling fixed points of g. Pick a basis $\{e_1, e_2, e_3$ with $e_1 \in x_-$, $e_2 \in x_o$, $e_3 \in x_+$, so that x_+ is the origin of the standard coordinate patch, and $T_{x_+}M$ is the x-axis (see Figure 4).

For any point $p = \begin{bmatrix} x \\ y \end{bmatrix}$ with $x \neq 0$,

$$\begin{aligned}
\lim_{n\to\infty} \frac{\log d(g^n \cdot p, x_+)}{n} &= \lim_{n\to\infty} \frac{\log[(\lambda_2\lambda_1^{-1})^{2n}x^2 + (\lambda_3\lambda_1^{-1})^{2n}y^2]}{2n} \\
&= \lim_{n\to\infty} \frac{\log[(\lambda_2\lambda_1^{-1})^{2n}(x^2 + \lambda_3\lambda_2^{-1}y^2)]}{2n} \\
&= \lim_{n\to\infty} \frac{2n\log[\lambda_2\lambda_1^{-1}] + \log[x + (\lambda_3\lambda_2^{-1})^{2n}y^2]}{2n} \quad = l_2 - l_1. \\
\lim_{n\to\infty} \frac{\log d(g^n \cdot p, T_{x_+}M)}{n} &= \lim_{n\to\infty} \frac{\log\,(\lambda_3\lambda_1^{-1})^n y}{n} \quad = l_3 - l_1. \\
\lim_{n\to\infty} \frac{\log d(g^n \cdot p, T_{x_+}M)}{\log d(g^n \cdot p, x_+)} &= \frac{l_1 - l_3}{l_1 - l_2} = \alpha_g
\end{aligned}$$

$$\lim_{n\to\infty} \frac{\log d(g^n \cdot p, T_{x_+}M)}{\log d(g^n \cdot p, x_+)^{\alpha_g}} = 1$$

$$\lim_{n\to\infty} \frac{d(g^n \cdot p, T_{x_+}M)}{d(g^n \cdot p, x_+)^{\alpha_g}} = 1$$

Thus, if $\alpha > \alpha_g$, the last limit would be infinite, so the Hölder exponent of M can be no larger than α_g. Note that, while we computed the Hölder exponent in a coordinate patch not containing all of M, the conclusion is projectively invariant for a compact neighborhood of x_+, and therefore for all of M. $\square$

Corollary 2.17. *Let M be a curve with projective automorphism group G. Then,*

$$\alpha_M \leq \alpha_G = \inf_{g \in G,\, g \text{ hyperbolic}} \alpha_g.$$

Definition 2.18. Let $\Omega \subset \mathbb{RP}^n$ be a projective domain. Define

$$\alpha_\Omega := \sup \{\alpha \mid \partial\Omega \text{ is } \alpha\text{-Hölder}\}.$$

The relationship between the automorphism group and Hölder exponent established in Corollary 2.17 carries over to projective domains.

2.4 β-Convexity

β-convexity is a useful notion dual to α-Hölder continuity, and is used in key proofs concerning convex domains.

Definition 2.19. Let $M \subset E$ be a manifold in an affine space E with a metric δ. Then M is *β-convex* if for all compact $K \subset M$ there is a constant C_K such that for all $x, y \in K$,

$$\delta(x, T_y M) \geq C_K \delta(x, y)^\beta,$$

where $T_y M$ is the tangent hyperplane to M at y.

As with Hölder continuity, given a projective domain $\Omega \subset \mathbb{RP}^n$, define

$$\beta_\Omega = \inf \{\beta \mid \partial\Omega \text{ is } \beta\text{-convex}\}.$$

Recall that the dual of a cone $U \subset \mathbb{R}^n$ is the set of linear functionals positive on U. Viewing a projective domain $\Omega \subset \mathbb{RP}^{n-1}$ as a cone in $\mathbb{R}^n$ we have a notion of a dual projective domain Ω^*. This relates Hölder continuity and β-convexity:

Lemma 2.20. (Benoist, [1])

$$\frac{1}{\alpha_\Omega} + \frac{1}{\beta_{\Omega^*}} = 1.$$

Definition 2.21. Fixing an $r_0 > 0$, we define a function $\phi^\delta_\beta : \Omega \to \mathbb{R}$ as:

$$\phi^\delta_\beta(x) := \frac{\inf\{\delta(x, y) | d_H(x, y) = r_0\}}{\sup\{\delta(x, y)^\beta | d_H(x, y) = r_0\}},$$

where d_H is the Hilbert metric on Ω. The idea is to compare the lengths of the major and minor axes of Hilbert-metric balls in the ambient metric δ.

Lemma 2.22. (Proposition 15 of [6]) *Let $\Omega \subset E \subset \mathbb{RP}^2$ be a strictly convex domain in an affine patch and $\beta \geq 2$. Then $\partial\Omega$ is β-convex if and only if ϕ^δ_β is bounded away from 0 on Ω for δ the Euclidean metric on E.*

2.5 Guichard's Theorem

Since the Hilbert metric is ultimately defined in terms of $\partial\Omega$, it is reasonable for the properties of $\partial\Omega$ to be related to those of the metric balls in the Hilbert metric on Ω. These can be connected by allowing a circle to approach the boundary of Ω and noting that in the limit the Hölder continuity and β-convexity of the metric ball must equal those of $\partial\Omega$. One can then analyze the action of the group elements on metric balls in Ω to calculate their β-convexity and, in turn, that of $\partial\Omega$. Below, we elaborate on this approach, taken by Guichard in [6] to prove:

Theorem 2.23. (Guichard [6]) *Let Ω be a strictly convex quasi-homogeneous domain in $\mathbb{RP}^n$ divided by a torsion-free group G. Then the following equivalent assertions are true:*

$$\alpha_\Omega = \alpha_G, \ \beta_\Omega = \beta_G$$

Proof. We provide a sketch of the proof.

The statements are equivalent by Lemma 2.20 and the same duality for groups dividing the dual domains. Guichard's proof focuses on the claim that $\beta_\Omega = \beta_G$.

By Lemma 2.22, β-convexity of $\partial\Omega$ can be verified by bounding the function ϕ^δ_β away from 0 on Ω.

To prove the claim, Guichard first defines a notion of (r, ϵ)-loxodromic elements, which act on $\mathbb{RP}^n$ and its exterior algebra in a controlled way. Let $G_{r,\epsilon}$ be the subset of (r, ϵ)-loxodromic elements of G. Guichard shows that for a given (r, ϵ), there exists a compact subset K_r of Ω such that $G_{r,\epsilon}K_r = \Omega$. The compactness of K_r assures that ϕ^δ_β is bounded away from 0 on it for any β. To show that this remains true for other points of Ω, Guichard writes an arbitrary $y \in \Omega$ as

$$y = gx, \ x \in K_r, \ g \in G_{r,\epsilon}.$$

He then analyzes the difference between $\phi^\delta_\beta(x)$ and $\phi^\delta_\beta(y)$. Assuming that $\beta \geq \beta_g$ for each g, he finds a uniform bound for the ratio of the two numbers. Thus, for $\beta > \beta_G$, the values of ϕ^δ_β away from the compact K_r are bounded away from 0, proving that $\partial\Omega$ is β-convex. □

Note that any group dividing Ω is virtually torsion-free by Selberg's Lemma in [13]. Extending the definition of α_g by setting $\alpha_g = \infty$ if g is torsion, Theorem 2.23 holds for arbitrary G (note that if G were entirely torsion, it would be finite and hence not act cocompactly on Ω). In particular, the theorem is immediately applicable to the Kac-Vinberg examples, where G is a Coxeter group.

3 The Generalized Tits Cone

We now generalize the construction of the Tits cone, finding a one-parameter family of cones for each hyperbolic triangle group. The constructed cones include the Kac-

Vinberg cones. Applying Guichard's result to the deformation space of cones leads to new conjectures.

3.1 Normalizing Triangle Group Representations

We first normalize the Kac-Vinberg cones of Section 2.2, as well as other triangle group representations.

Let W be an irreducible hyperbolic triangle group:

$$W = \left\langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^p = (s_1 s_3)^q = (s_2 s_3)^r = 1 \right\rangle, 2 < p \leq q \leq r, \frac{1}{p}+\frac{1}{q}+\frac{1}{r} < 1.$$

Let $\rho : W \to \mathrm{GL}(\mathbb{R}^3)$ be a faithful representation of W such that $R_i = \rho(s_i)$ is a reflection for each $i = 1, 2, 3$. Let L_i be the fixed plane, and l_i the -1-eigenspace of R_i for each i. For each R_i, R_j, the corresponding fixed lines L_i, L_j split $\mathbb{R}^3$ into quadrants. We require furthermore that one of these be a fundamental domain for the action of the group $\langle R_i, R_j \rangle$.

We now define a normalized form for $G = \rho(W)$.

Lemma 3.1. *There is a basis for $\mathbb{R}^3$ such that R_1 and R_2 are Euclidean reflections given by:*

$$R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_2 = \begin{bmatrix} \cos\frac{2\pi}{p} & \sin\frac{2\pi}{p} & 0 \\ \sin\frac{2\pi}{p} & -\cos\frac{2\pi}{p} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Proof. Let $(\cdot,\cdot)$ denote the standard inner product on $\mathbb{R}^3$, and define a new inner product B by

$$B(x, y) = \sum_{g \in \langle R_1, R_2 \rangle} (gx, gy)$$

This bilinear form is still positive-definite and non-degenerate, and also invariant under the group $\langle R_1, R_2 \rangle$. We adopt it as the inner product on $\mathbb{R}^3$. Let e_3 be a unit vector in $L_1 \cap L_2$ and complete $\{e_3\}$ to an orthonormal basis of $\mathbb{R}^3$. Since the reflections fix e_3, in this basis they must be of the form

$$R_i = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ e & f & 1 \end{bmatrix}$$

Furthermore, they must preserve the inner product B. This is equivalent to R_1 and R_2 being symmetric:

$$R_i = \begin{bmatrix} a & b & 0 \\ b & d & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note also that each R_i is a reflection, so in each case of trace 1, so $a = -d$.

Restricting to the plane spanned by the first two basis vectors, we get that R_1 and R_2 are orthogonal (Euclidean) reflections in $\mathbb{R}^2$ fixing the origin. Up to conjugation by a rotation, R_1 fixes (in cylindrical coordinates) the plane $\theta = 0$, and R_2 fixes $\theta = \pm\frac{\pi}{p}$; we may assume the sign is positive. Such transformations are unique and have the stated matrix representation. $\square$

Definition 3.2. Using the basis defined in Lemma 3.1, we use the projection

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x/z \\ y/z \end{bmatrix}$$

to view the invariant planes L_i in the standard coordinate patch of $\mathbb{RP}^2$. After projection, L_1, L_2, and L_3 become lines. Now, if L_3 intersects L_2 inside the coordinate patch, we rescale the patch so that the Euclidean distance from the origin to the intersection is 1. We then denote by d the x-coordinate of $L_1 \cap L_3$. This may be any number including ∞. In the case that the rescaling was impossible, we set $d = 0$.

Given this normalization, we say that ρ is of characteristic (p, q, r, d). Up to the choice of ordered generators (a marking), a representation has a unique characteristic.

3.2 Constructing the Generalized Tits Cone

We now construct a generalized Tits cone for a representation of characteristic (p, q, r, d).

Definition 3.3. Let W be an irreducible hyperbolic triangle group and $d \in (\cos\frac{\pi}{p}, \sec\frac{\pi}{p})$. Define the following reflections:

$$R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 = \begin{bmatrix} \cos\frac{2\pi}{p} & \sin\frac{2\pi}{p} & 0 \\ \sin\frac{2\pi}{p} & -\cos\frac{2\pi}{p} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

These reflections satisfy $(R_1R_2)^p = I$. We define the third reflection R_3 by specifying its eigenvectors (recall that a reflection has eigenvalues $\{1, 1, -1\}$). We first require that

$$R_3 \begin{bmatrix} d \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} d \\ 0 \\ 1 \end{bmatrix}, \text{ and } R_3 \begin{bmatrix} \cos\frac{\pi}{p} \\ \sin\frac{\pi}{p} \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{p} \\ \sin\frac{\pi}{p} \\ 1 \end{bmatrix},$$

i.e., R_3 fixes the plane spanned by these two vectors. Let the third eigenvector be $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, define a change-of-basis matrix M, and impose the condition that the three eigenvectors are linearly independent:

$$M = \begin{bmatrix} d & \cos\frac{\pi}{p} & x \\ 0 & \sin\frac{\pi}{p} & y \\ 1 & 1 & z \end{bmatrix}, \ \det M = 1$$

R_3 is then given by

$$R_3 = M \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} M^{-1}.$$

Furthermore, to satisfy the triangle group relations, require:

$$(R_1R_3)^q = I \longleftrightarrow \operatorname{tr}(R_1R_3) = 2\cos\frac{2\pi}{q} + 1$$

$$(R_2R_3)^r = I \longleftrightarrow \operatorname{tr}(R_2R_3) = 2\cos\frac{2\pi}{r} + 1$$

Together with $\det M = 1$, this gives three linear equations in x, y, z. When

$$d \notin \{\cos\frac{\pi}{p}, \sec\frac{\pi}{p}\},$$

we can solve for $\{x, y, z\}$ and complete the definition of R_3. We then set $\rho(s_i) = R_i$. This gives a representation of the triangle group in $GL(3, \mathbb{R})$. We will show (Corollary 3.8) that this representation is faithful, so ρ has characteristic (p, q, r, d).

Following the notation of the Tits cone, define C to be the cone

$$C = \mathbb{R}^+ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \oplus \mathbb{R}^+ \begin{bmatrix} \cos\frac{\pi}{p} \\ \sin\frac{\pi}{p} \\ 1 \end{bmatrix} \oplus \mathbb{R}^+ \begin{bmatrix} d \\ 0 \\ 1 \end{bmatrix}.$$

For each i, consider the complement of L_i, the $+1$-eigenspace of R_i, and let A_i be the component containing C (e.g., A_1 is the half-space $y > 0$).

Lastly, define

$$G = \rho(W), \ \ U = \bigcup_{g \in G} g\overline{C}.$$

The constraints on d ensure that the corners of the triangular cone C look like Figure 2:

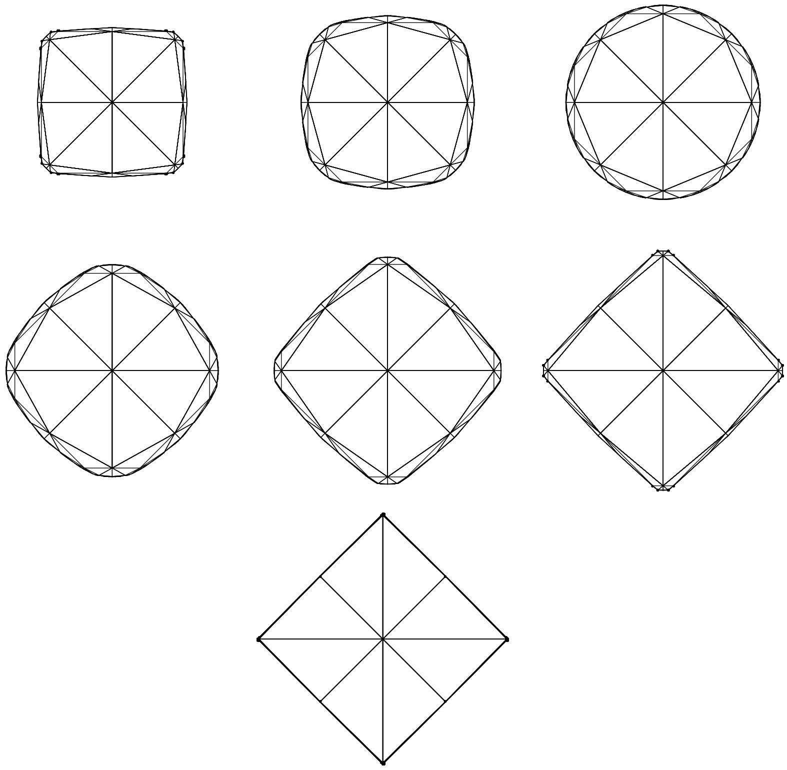

Figure 5: Generalized Tits cone for the (4,4,4) triangle group with d increasing from .8 to 1.4 in increments of .1.

Lemma 3.4. *In the above notation, for distinct* $i, j \in \{1,2,3\}$*,* $n = |R_iR_j|$*, there exists a basis of* $\mathbb{R}^3$ *such that*

$$R_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } R_j = \begin{bmatrix} \cos\frac{2\pi}{n} & \sin\frac{2\pi}{n} & 0 \\ \sin\frac{2\pi}{n} & -\cos\frac{2\pi}{n} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Furthermore,

$$A_i \cap A_j = \mathbb{R}^+ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \oplus \mathbb{R}^+ \begin{bmatrix} \cos\frac{\pi}{p} \\ \sin\frac{\pi}{p} \\ 1 \end{bmatrix} \oplus \mathbb{R} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Proof. The lemma is true by construction for $i = 1, j = 2$. We focus on the case $i = 1, j = 3$.

Lemma 3.1 shows that R_i, R_j can be written in the desired form.

We now approach the normalization in a different way to prove the second assertion. We first conjugate by the affine translation $\begin{bmatrix} 1 & 0 & -d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ so that R_3 now fixes $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Note that R_1 remains unchanged.

Say M is a matrix that conjugates the new R_1 and R_3 to the desired form. Then M must commute with R_1 and have $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ as an eigenvector (since M $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ has to be fixed by R_3 and also stay in the fixed eigenspace of R_1. This gives us that M must preserve the planes

$$\operatorname{span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right), \operatorname{span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right),$$

These divide $\mathbb{R}^3$ into quadrants, which M must permute.

Now, since $d > \cos\frac{\pi}{p}$, we know that $A_1 \cap A_3$ is contained in one of these quadrants. This remains true for $M(A_1 \cap A_3)$. Given R_1 and R_3 in normalized form, there are four choices for $A_1 \cap A_3$. Two of these are not contained in a quadrant, and the other two differ by an order-2 rotation that commutes with the reflections. We may pick the one we want.

The other cases follow immediately, with the use of $d < \sec\frac{\pi}{p}$ to prove the cases $i, j \in \{2,3\}$. $\square$

Corollary 3.5. *The construction of the Tits cone fails for* $d \notin (\cos\frac{\pi}{p}, \sec\frac{\pi}{p})$*.*

Proof. While we can construct cones outside this interval, the corners are not acute and the corresponding space U does not have nice combinatorial properties. In particular, the quotient space U/G is an orbifold if and only if d is in the specified range. □

Corollary 3.6. *The generalized Tits cone with the origin removed, $U\backslash\{0\}$, is open.*

Note that we have excluded the case of ideal hyperbolic triangle groups, where R_iR_j is of infinite order for some i, j from this discussion. In these cases, $U\backslash\{0\}$ is not open and the theory of quasi-homogeneous domains does not apply.

Theorem 3.7. *We use the above notation for the generalized Tits cone. Let $w \in W$. Then*

$$\begin{aligned}&\text{(P) For each } w \in W \text{ and } s \in S, \text{ either}\\ &\quad wC \subset A_s \text{ and } l(w) = l(sw) - 1, \text{ or}\\ &\quad wC \subset sA_s \text{ and } l(w) = l(sw) + 1.\end{aligned}$$

Informally, if wC is on the "negative" side of the cone with respect to s, then there is a minimal word equivalent to w that starts with s.

Proof. Lemma 3.4 and Lemma 2.2 give us Property (P) for each group W_{ij}. The proof for W is then identical to Theorem 2.4, except for the use of Lemma 3.4 instead of Lemma 2.3. □

Corollary 3.8. *The representation of W given by $\rho(s_i) = R_i$ is faithful, and that the generalized Tits cone is convex. The proof is the same as in Section 2.1 for the standard Tits cone.*

The proof of Theorem 2.7, which states that the Tits cone contains no full lines, relies on the bilinear form on the Tits cone. However, a generalized Tits cone is given by an symmetric bilinear form on $\mathbb{R}^3$ only if it is an elliptical cone. We must therefore reprove the result for the generalized case. We first define some notation and prove a lemma:

Definition 3.9. For a generalized Tits cone U, Define Ω to be the projection of $U\backslash\{0\}$ onto the sphere S^2, and E the upper hemisphere $z > 0$ of S^2.

Lemma 3.10. *Let U_1, U_2 be two generalized Tits cones for a given triangle group (the last parameter, d, may vary). Then the corresponding spaces Ω_1 and Ω_2 are homeomorphic.*

Proof. The closures of the generating cones $\overline{C}$ project to triangles on S^2, which are homeomorphic by stretching the side of length d and imposing a linear isomophism on the rest of the simplex. Since the rest of Ω_i is defined by the faithful G-action, the homeomorphism of triangles extends to an homeomorphism of the spaces Ω_i. □

Theorem 3.11. *Let G be a hyperbolic triangle group. Then the closure of the generalized Tits cone $\overline{U}$ contains no full lines.*

Proof. If $\overline{\Omega} \subset E$, then we are done, so assume otherwise.

First, consider the case where Ω intersects the equator of S^2, so there is a point $p \in \overline{\Omega} \cap \partial E$. Since Ω is open, it contains an open neighborhood around p, so there is a point q of Ω that is in the lower hemisphere. The action of $\langle R_1, R_2 \rangle$ provides more points in the lower hemisphere. Since $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in \Omega$, we have by convexity that the generalized Tits cone $U = \mathbb{R}^3$ and $\Omega = S^2$. However, since G is hyperbolic, its Tits cone does not contain a line by Theorem 2.7, so its projection Ω' into S^2 lies entirely in the upper hemisphere E and is contractible. By Lemma 3.10, Ω is homeomorphic to Ω' is impossible.

The remaining case is that Ω is contained entirely in E but has a point in the closure of E. By applying the reflections R_1, R_2 to the point and invoking convexity, $\overline{\Omega} = \overline{E}$, so $\Omega = E$. Since G leaves Ω invariant, it must also leave $\partial\Omega$ invariant, so all elements of G are in the affine group Aff(2,$\mathbb{R}$). The Tits cone provides a faithful linear representation of G, so G is virtually torsion free by Selberg's Lemma ([13]), i.e. there is a torsion-free $G' \subset G$ of finite index. Since G acts properly, so does G', and thus Ω/G' is a manifold, with an induced affine structure. We can also find a hyperbolic structure on Ω/G' by means of the G-equivariant homeomorphism (Lemma 3.10) between Ω and the corresponding Ω' for the Tits cone. Since Ω/G' has a hyperbolic structure, it has genus greater than one. However, Benzecri showed in [2] that there is no affine structure on such a manifold. $\square$

Consider the projection Ω of U into S^2 (or, equivalently, into $\mathbb{RP}^2$). We may now apply Theorems 1.1, 5.1 and Fact 5.4 of [1].

Corollary 3.12. *Since W is a hyperbolic group,*

1. *Ω is strictly convex (i.e. its boundary does not contain a segment).*
2. *$\partial\Omega$ is once differentiable.*
3. *All non-torsion elements of G are hyperbolic (have distinct real eigenvalues).*
4. *The action of G on $\partial\Omega$ is minimal.*
5. *The set $\{$ fixed boundary points of $g \mid g \in G$ non-torsion$\}$ is dense in $\partial\Omega \times \partial\Omega$.*

Thus, for a given hyperbolic triangle group W, we have constructed a deformation space of cones with the same combinatorial properties as the Tits cone. We now apply this to the study of Hölder regularity.

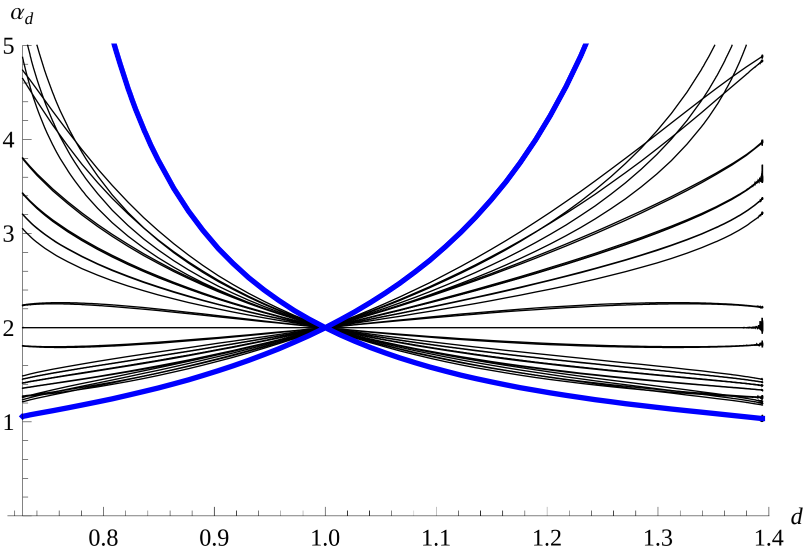

Figure 6: α_g for various g of length at most 10, as the parameter d for the generalized Tits cone varies. p=q=r=4. The thick graphs correspond to the Coxeter elements $R_1R_2R_3$ and $R_3R_2R_1$.

3.3 Some Conjectures

Since for a hyperbolic group W a generalized Tits cone U is convex, we may apply Guichard's Theorem 2.23. Denote by Ω the projectivization of U, i.e. the image of $U \backslash \{0\}$ under the projection map

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x/z \\ y/z \end{bmatrix},$$

viewed as a coordinate patch of $\mathbb{RP}^2$. Since U is strictly convex, the map is defined on all of $U \backslash \{0\}$. By Theorem 2.23 the Hölder coefficient of $\partial\Omega$ equals the infimum of the Guichard alphas for the non-torsion elements of G:

$$\alpha_\Omega = \inf_{g \in G, |g| = \infty} \alpha_g = \inf_{g \in G, |g| = \infty} \frac{l_3 - l_1}{l_3 - l_2},$$

where $l_1 > l_2 > l_3$ are the logs of the absolute values of the eigenvectors of g.

The explicit construction of the generalized Tits cones allowed us to apply computational methods to the calculation of Hölder continuity, leading to the three conjectures.

Figure 6 shows the graphs for α_g of various elements of the (4,4,4) triangle group as the deformation parameter d changes. The highlighted curves represent the Coxeter elements. Based on this graph (and similar further experimental data), we make the following conjectures:

Conjecture 3.13. *In the notation for the generalized Tits cone,*

$$\alpha_\Omega = \min\{\alpha_{R_1R_2R_3}, \alpha_{R_3R_2R_1}\}$$

Attempts to prove the conjecture are complicated by the infinite (or at least prohibitively large) number of equivalence classes of elements, up to conjugacy and powers. Furthermore, the graph shows that the α_g lines intersect at multiple points, so a monotonicity result cannot be established.

Conjecture 3.14. *Using the notation for the generalized Tits cone with $p = 4, q = 4, r = 4$, there exists $d \in (\cos\frac{\pi}{p}, \sec\frac{\pi}{p})$ such that $\alpha_{R_1R_2R_3} = \alpha$ for any $\alpha \in (1,2)$.*

Corollary 3.15. *Let $\alpha \in (1,2)$. There exist quasi-homogeneous domains Ω_1, Ω_2 with*

$$\alpha_{\Omega_1} < \alpha, \quad \alpha_{\Omega_2} > \alpha.$$

If Conjecture 3.13 is true, Ω_1 and Ω_2 can be chosen so that these are equalities.

Proof. Assuming Conjecture 3.14, we find the Tits cone U and corresponding projective domain Ω such that $\alpha_{R_1R_2R_3} = \alpha$. The first part of the conjecture is then given by Guichard's Theorem:

$$\alpha_\Omega = \inf_{g \in G,\, |g| = \infty} \alpha_g \leq \alpha_{R_1R_2R_3}$$

The second part is given by duality:

$$\frac{1}{\alpha_\Omega} + \frac{1}{\alpha_{\Omega^*}} = 1.$$

If $\alpha_\Omega = \min\{\alpha_{R_1R_2R_3}, \alpha_{R_3R_2R_1}\}$, then it is continuous as a function of d. Since it can be arbitrarily close to 1 and 2, it can be anywhere in the range. $\square$

4 Coxeter Elements

We conclude by proving two important properties of Coxeter elements in hyperbolic triangle groups. We also note that Coxeter elements are critical in other situations, such as the proof of the Goldman-Parker conjecture ([12, 5]) which considers representations of ideal triangle groups in the isometries of complex hyperbolic space.

4.1 Infinite Order, Essential

We now prove Theorem 1.7.

Recall that hyperbolic automorphisms of $\mathbb{H}$ have infinite order and have trace greater than two. To prove Theorem 1.7, we will show that there is an action of a triangle group on the upper half-plane $\mathbb{H}_u$ such that the square of a Coxeter element acts by a hyperbolic transformation.

The Tits cone construction shows that W can be embedded into Isom($\mathbb{H}$) as a group generated by reflections. We consider a faithful representation of W into the isometries of $\mathbb{H}_u$:

$$G = \left\langle R_1, R_2, R_3 \;\middle|\; (R_1R_2)^p = (R_2R_3)^q = (R_1R_3)^r = R_i^2 = I \right\rangle \subset \text{ PSL(2,}\mathbb{R})$$

Lemma 4.1. *Set $A = R_1R_2$, $B = R_2R_3$, $AB = R_1R_3$. Then the trace of the square of a Coxeter element,*

$$tr(ABA^{-1}B^{-1}) = tr(R_1R_3R_2R_1R_3R_2) = tr(R_1R_3R_2)^2,$$

is given by Goldman's favorite polynomial $K(x, y, z)$:

$$tr(ABA^{-1}B^{-1}) = K(x, y, z) = x^2 + y^2 + z^2 - xyz - 2$$

where $x = tr(A), y = tr(B), z = tr(AB)$.

Proof. Given $X \in$ SL(2, $\mathbb{R}$), note that by the Cauchy-Schwartz Theorem (a matrix satisfies its minimal polynomial),

$$X^2 - \text{tr}(X)X + I = 0$$
$$X - \text{tr}(X)I + X^{-1} = 0$$

Multiplying through by another matrix $Y \in$ SL(2, $\mathbb{R}$),

$$XY - \text{tr}(X)Y + X^{-1}Y = 0$$

Taking traces,

$$\text{tr}(XY) - \text{tr}(X)\text{tr}(Y) + \text{tr}(X^{-1}Y) = 0$$
$$\text{tr}(XY) = \text{tr}(X)\text{tr}(Y) - \text{tr}(X^{-1}Y)$$

Also note that in SL(2,$\mathbb{R}$) trace is invariant under inversion as well as conjugation. We now apply the above trace expansion to $ABA^{-1}B^{-1}$, taking $X = A$ and $Y = BA^{-1}B^{-1}$:

$$\begin{aligned}\text{tr}(ABA^{-1}B^{-1}) &= \text{tr}(A)\text{tr}(BA^{-1}B^{-1}) - \text{tr}(A^{-1}BA^{-1}B^{-1}) \\ &= \text{tr}(A)^2 - \text{tr}(A^{-1}BA^{-1}B^{-1})\end{aligned}$$

Splitting again with $X = A^{-1}B$, $Y = A^{-1}B^{-1}$,

$$\begin{aligned}\text{tr}(ABA^{-1}B^{-1}) &= \text{tr}(A)^2 - [\text{tr}(A^{-1}B)\text{tr}(A^{-1}B^{-1}) - \text{tr}(B^{-1}AA^{-1}B^{-1})] \\ &= \text{tr}(A)^2 - \text{tr}(AB)[\text{tr}(A^{-1}B)] + \text{tr}(B^2) \\ &= \text{tr}(A)^2 - \text{tr}(AB)[\text{tr}(A)\text{tr}(B) - \text{tr}(AB)] + [\text{tr}(B)\text{tr}(B) - \text{tr}(I)] \\ &= \text{tr}(A)^2 + \text{tr}(AB)^2 - \text{tr}(A)\text{tr}(B)\text{tr}(AB) + \text{tr}(B)^2 - 2 \\ &= \text{tr}(A)^2 + \text{tr}(B)^2 + \text{tr}(AB)^2 - \text{tr}(A)\text{tr}(B)\text{tr}(AB) - 2 \\ &= x^2 + y^2 + z^2 - xyz - 2\end{aligned}$$

□

Now, A is the product of reflections in two lines meeting at angle $\frac{\pi}{p}$, so it is a rotation by angle $\frac{2\pi}{p}$. It must therefore have trace $\pm 2\cos(\frac{\pi}{p})$. Likewise, B has trace $\pm 2\cos(\frac{\pi}{q})$, and AB has trace $\pm 2\cos(\frac{\pi}{r})$. We are free to chose signs for two of these, but the third is determined once we make the choice. We choose $\operatorname{tr}(A) = -2\cos(\frac{\pi}{p})$, $\operatorname{tr}(B) = -2\cos(\frac{\pi}{q})$ and delay the proof (Lemma 4.3) that this forces $\operatorname{tr}(AB) = -2\cos(\frac{\pi}{q})$.

Lemma 4.2. *$K(x, y, z) > 2$ for a hyperbolic group.*

Proof. Making the above choices for the traces of A, B, and AB, Goldman's favorite polynomial becomes

$$K(p,q,r) = 4[(\cos(\frac{\pi}{p}))^2 + (\cos(\frac{\pi}{q}))^2 + (\cos(\frac{\pi}{r}))^2 + 2\cos(\frac{\pi}{p})(\cos(\frac{\pi}{q}))(\cos(\frac{\pi}{r})) - \frac{1}{2}]$$

We derive this from the cosine angle-sum formula. To ease the notation, define

$$P = \cos(\frac{\pi}{p}), Q = \cos(\frac{\pi}{q}), R = \cos(\pi - \frac{\pi}{p} - \frac{\pi}{q}) = -\cos(\frac{\pi}{p} + \frac{\pi}{q}).$$

If we at the same time had $R = \cos(\frac{\pi}{r})$, this would correspond to the Euclidean angle case.

$$\begin{aligned}
\cos\left(\frac{\pi}{p} + \frac{\pi}{q}\right) &= \cos\frac{\pi}{p}\cos\frac{\pi}{q} - \sin\frac{\pi}{p}\sin\frac{\pi}{q} \\
&= \cos\frac{\pi}{p}\cos\frac{\pi}{q} - \sqrt{1 - \cos^{\frac{2\pi}{p}}}\sqrt{1 - \cos^{\frac{2\pi}{q}}} \\
-R &= PQ - \sqrt{1-P^2}\sqrt{1-Q^2} \\
\sqrt{1-P^2}\sqrt{1-Q^2} &= PQ + R \\
(1-P^2)(1-Q^2) &= (PQ+R)^2 \\
1 - P^2 - Q^2 + P^2Q^2 &= P^2Q^2 + 2PQR + R^2 \\
P^2 + Q^2 + R^2 - 2PQR - \frac{1}{2} &= \frac{1}{2}
\end{aligned}$$

So that in the Euclidean case we would have $K(x, y, z) = 2$, either the identity or a parabolic element.

Now, since $\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} < \pi$, $\cos(\frac{\pi}{r}) > R$. We now want to prove the following statement:

$$P^2 + Q^2 + (\cos(\frac{\pi}{r}))^2 - 2PQ(\cos(\frac{\pi}{r})) - \frac{1}{2} > P^2 + Q^2 + R^2 - 2PQR - \frac{1}{2}$$

$$(\cos(\frac{\pi}{r}))^2 - 2PQ(\cos(\frac{\pi}{r})) > R^2 - 2PQR$$

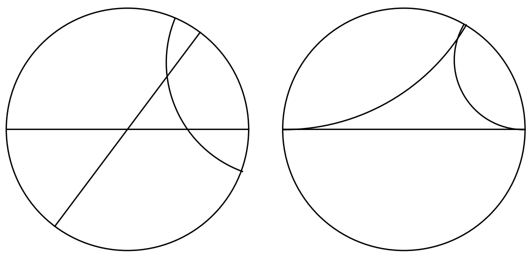

Figure 7: A triangle in the hyperbolic plane (left) can be deformed to the ideal case (right).

To do this, we show that $\cos(\theta)^2 - 2PQ(\cos(\theta))$ is decreasing with respect to θ on the interval $(\cos(\pi - \frac{\pi}{p} - \frac{\pi}{q}), 0)$. The derivative is negative on this interval since

$$-\sin(\theta)(\cos(\theta) - 2PQ) < 0 \text{ iff } (\cos(\theta) - 2PQ) > 0),$$

and we have

$$\cos(\theta) > \cos(\pi - \frac{\pi}{p} - \frac{\pi}{q}) - \cos(\frac{\pi}{p})\cos(\frac{\pi}{q}) =$$
$$-\cos(\frac{\pi}{p} + \frac{\pi}{q}) - \cos(\frac{\pi}{p})\cos(\frac{\pi}{q}) = \sin(\frac{\pi}{p})\sin(\frac{\pi}{p}) \geq 0$$

Thus, $\frac{\pi}{r} > \pi - \frac{\pi}{p} - \frac{\pi}{q}$ implies $\cos(\frac{\pi}{r}) > R$, which in turn implies

$$(\cos(\frac{\pi}{r}))^2 - 2PQ(\cos(\frac{\pi}{r})) > R^2 - 2PQR,$$

and we have

$$P^2 + Q^2 + R^2 - 2PQR - \frac{1}{2} > \frac{1}{2}.$$

□

Then, the trace of $ABA^{-1}B^{-1}$ is greater than two, so $ABA^{-1}B^{-1}$ must be a hyperbolic transformation. Up to conjugation, it is the square of a Coxeter element g, so g has infinite order. The inverse g^{-1} also has infinite order, and represents the other conjugacy class of Coxeter elements in the triangle group.

We still have to decide how to assign signs to the traces of A, B, and AB.

Lemma 4.3. *If $tr(A) = -2\cos(\frac{\pi}{p})$, $tr(B) = -2\cos(\frac{\pi}{q})$, and $tr(AB) = \pm\cos(\frac{\pi}{r})$, then $tr(AB) = \cos(\frac{\pi}{r})$.*

Proof. We first consider the limiting case $p = \infty$, $q = \infty$, $r = \infty$. A, B, AB are then parabolic elements. Without loss of generality,

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ b & -1 \end{bmatrix}$$

The product is then

$$AB = \begin{bmatrix} 1+b & -1 \\ -b & 1 \end{bmatrix},$$

which has trace $2 + b = \pm\cos(\pi/\infty) = \pm 2$.

If we choose $\mathrm{tr}(AB) = 2$, then $b = 0$, making B projectively equivalent to the identity. Since it's the product of two different reflections, this is impossible. We must therefore have a negative trace.

Now, recall that $A = R_1R_2$, $B = R_2R_3$, where R_1, R_2, R_3 are reflections over hyperbolic lines defining a triangle with interior angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$ (Figure 7). If we move the lines, the corresponding transformations vary continuously. If we move them so that the endpoints of the triangle move toward the boundary, the interior angles decrease, so the trace of AB approaches ± 2 through a monotone function. Since the interior angle starts off acute, it stays acute, so the trace of AB is never 0. Thus, $\mathrm{tr}(AB) < 0$ in the case we are interested in. $\square$

4.2 Minimizing Translation Length in the Hilbert Metric

Lemma 4.4. (McMullen, [11]) *Let Ω be a strictly convex domain in $\mathbb{RP}^2$, and $g \in PGL(3, \mathbb{R})$ an automorphism of Ω with eigenvalues $|\lambda_1| > |\lambda_2| > |\lambda_3|$ and $l_i = \log(|\lambda_i|)$. Then,*

$$\inf_{p\in\Omega} d_H(p, gp) = \frac{l_1 - l_3}{2}$$

Proof. Let $x_+, x_o, x_- \in \mathbb{RP}^2$ be, respectively, the attracting, saddle, and repelling fixed points of g. Pick a basis $\{e_1, e_2, e_3$ with $e_1 \in x_-$, $e_2 \in x_o$, $e_3 \in x_+$, so that x_+ is the origin of the standard coordinate patch, and $T_{x_+}M$ is the x-axis (as in Figure 4). g acts on the coordinate patch by the linear transformation $\begin{bmatrix} \lambda_2\lambda_1^{-1} & 0 \\ 0 & \lambda_3\lambda_1^{-1} \end{bmatrix}$.

We let $p = \begin{bmatrix} 0 \\ y \end{bmatrix}$ and calculate the translation length (restricting to the positive y-axis for the second equality):

$$\begin{aligned} d(p, gp) &= d\left(\begin{bmatrix} 0 \\ y \end{bmatrix}, \begin{bmatrix} 0 \\ \lambda_3\lambda_1^{-1}y \end{bmatrix}\right) \\ &= \frac{1}{2}\log[\infty, y; \lambda_3\lambda_1^{-1}y, 0] \\ &= \frac{1}{2}\log\left|\frac{(y-0)(\lambda_3\lambda_1^{-1}y - \infty)}{(y-\infty)(\lambda_3\lambda_1^{-1}y - 0)}\right| \\ &= \frac{1}{2}\log\left|\frac{y}{-\lambda_3\lambda_1^{-1}y}\right| \\ &= \frac{l_1 - l_3}{2} \end{aligned}$$

Note that the above calculations technically require taking limits instead of dividing infinities, but the result is the same.

Now, if p is not on the y-axis, we may project segment $\overleftrightarrow{p, gp} \cap \Omega$ into the positive y-axis. Projections decrease the value of the cross-ratio, and thus the distance between p and gp is at least as large as if p had been on the y-axis in the first place. Thus, p is moved by a distance of at least $\frac{l_1 - l_3}{2}$. $\square$

Theorem 4.5. (McMullen, [11]) *Coxeter elements minimize translation length in the generalized Tits cone Ω among the essential elements of W.*

$$\inf_{w \in W, w \text{ essential}} \left(\inf_{p \in \Omega} d_H(p, gp) \right) = \min_{w \in W, w \text{ Coxeter}} \left(\inf_{p \in \Omega} d_H(p, gp) \right)$$

Proof. We say a curve in Ω represents an element $w \in W$ if it connects a point p with its image wp. The translation length of an element is the infimum over the lengths of all curves γ that represent it. Since the geodesics in the Hilbert metric are the straight segments, we may restrict the infimum to the γ that are segments.

We now show that any straight segment γ representing w can be modified without changing its length to represent a Coxeter element.

Pick a basepoint p, which is without loss of generality contained in C. Consider the straight segment γ connecting p and wp. Using the inductive process in Corollary 2.6, we may build a minimal representation $s_{i_1} \dots s_{i_k}$ of w such that a γ goes from p to wp, first crosses the side of the fundamental region corresponding to s_{i_1}, then $s_{i_1} s_{i_2}$, etc.

Now, since w is essential, all generators of W must appear in $s_{i_1} \dots s_{i_k}$, so some Coxeter element g appears as a subword of w. These combinatorial properties of the group are reflected in the generalized Tits cone.

We now modify γ: it is only allowed to cross a wall (of the image of the fundamental domain) when the wall corresponds to a letter of the Coxeter element we are trying to create. If it doesn't, γ reflects off of the wall (the modification to γ can be made precise by considering orbifold covering spaces). For example, if $w = s_1 s_3 s_1 s_2$, we would allow it to cross the side corresponding to s_1 only once. As a result, the modified γ now represents a Coxeter element, but has the same length. $\square$

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