Tangent Approximations to Sub-Riemannian Manifolds

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Abstract

A sub-Riemannian manifold models constrained motion through a choice of a "horizontal distribution" on the tangent bundle. The standard definitions of tangent space and the differential of a smooth map break down in this setting. Following papers by Bellaïche and Ponge, I will discuss the way these notions are replaced by talking about non-abelian vector spaces (Carnot groups) and induced maps between them.

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1 General Definitions and Examples

1.1 Sub-Riemannian Manifolds and Connectivity

Definition 1. A sub-Riemannian structure on a manifold M is a distribution H on TM (not necessarily of constant rank) with a smooth function g on TM that restricts to a Riemannian metric on H and takes the value ∞ away from H.

Locally, we may choose, for k = maxiumum rank of H, vector fields X_i such that $H = \langle X_1, \ldots, X_n \rangle$. Since I will discuss local properties, I will assume that this chioce is global.

Definition 2. A path $\gamma : I \to M$ is horizontal (or controlled) if $\gamma'(t) \in H_{\gamma(t)}$ for all t. In this case, γ solves a control equation

$$\gamma'(t) = \sum a_i(\gamma(t))X_i(\gamma(t))$$

for some functions a_i on M. Solutions to such equations are interesting to control theory, where a process is controlled by means of some vector fields X_i and the resulting paths in the state space M are of interest. The non-constant rank of H in the definition above is due to demands from control theory.

Definition 3. For a horizontal path, we may define

$$L(\gamma) = \int_I |\gamma'(t)|_g \, dt = \int_I \sqrt{\sum a_i^2(t)} dt$$

$$d_{CC}(p,q) = \inf_{\gamma : p \text{ to } q} L(\gamma), \text{ or } \infty \text{ if no horizontal path exists}$$

The metric d_{CC} is called the Carnot-Carathèodory metric.

Theorem 4. *(Sussman 1973) Let $E_p = \{q \in M \mid d(p,q) < \infty\}$. Then for all $p \in M$, E_p is an embedded submanifold.*

In general this provides a foliation on M. Note that in the case that $\mathbb{H}$ is closed under the Lie bracket, the E_p are integral submanifolds for H. In general, they are integral manifolds for the Lie closure of H.

Definition 5. Let $[H]^0 = H, [H]^n = [H, [H]^{n-1}]$. If the span of $\{[H]^n\}_{n=0}^{\infty}$ is all of TM, then H:

1. is completely non-integrable
2. is bracket-generating
3. satisfies Hormander's (or Chow's) condition

All these are equivalent names for the property above.

Theorem 6. *(Chow 1939) If H is bracket-generating, then there is a horizontal path between any two points of M. Furthermore, d_{CC} induces the standard topology on M.*

1.2 Examples

Standard and degenerate Riemannian structures are sub-Riemannian. The basic degenerate example is the Grushin plane:

1.2.1 Grushin Plane

Take $M = \mathbb{R}^2, H = \left\langle X_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, X_2 = \begin{bmatrix} 0 \\ x \end{bmatrix} \right\rangle$, and the metric making the two vector fields orthonormal.

1. The metric is Riemannian away from the y-axis. Paths perpendicular to the y-axis are horizontal.
2. $[X_1, X_2] = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so H is bracket-generating.
3. Figure 1 shows geodesics between various points in the Grushin plane. Note that there are many "sinusoidal" ways to get from one point on the y-axis to another. Furthermore, a curve leaving a point on the y-axis is not determined by its direction. While the rank(H)=1, for each point in $\mathbb{R}^2$ there is a geodesic from the origin that reaches the point at time 1.

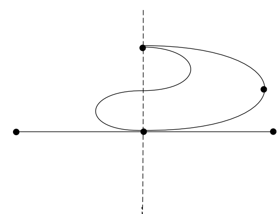

Figure 1: Grushin Plane Geodesics

4. The dilation $\delta_t : (x, y) \mapsto (tx, t^2 y)$ leaves H invariant and scales d_{CC} by a factor of t. Sub-Riemannian spaces with dilations will be useful to us later.
5. Lastly, note that the metric away from the y-axis does not agree with the Euclidean metric, and that the Euclidean metric restricted to H does not give a sub-Riemannian metric since it doesn't extend correctly to the tangent bundle.

1.2.2 CR Structures: boundaries of complex domains

Sub-Riemannian structures are relevant to complex analysis because of the connection to tangent spaces of boundaries of domains.

Let $\Omega \in \mathbb{C}^n$ be a domain and $p \in \partial\Omega$. Then a standard tangent space $T_p^{\mathbb{R}}\partial\Omega$ to the boundary has codimension one and does not carry a complex vector-space structure. This makes it unnatural as a complex-analytic object. Instead, one can consider the largest complex vector space contained in the $T_{\mathbb{R}}\partial\Omega$:

$$T_p^{\mathbb{C}}\partial\Omega = T_p^{\mathbb{R}}\partial\Omega \cap \mathfrak{i}T_p^{\mathbb{R}}\partial\Omega$$

We then get $T^{\mathbb{C}}\partial\Omega \subset T^{\mathbb{R}}\partial\Omega$ a codimension-1 subbundle. The resulting structure is called a CR (Cauchy-Riemann) structure and becomes a sub-Riemannian structure with a choice of g.

This construction also provides a connection to complex hyperbolic geometry: complex hyperbolic n-space can be modeled on the interior of a unit ball in $\mathbb{C}^n$, and one class of totally geodesic hyperplanes provides horizontal submanifolds on the boundary S^{n-1}.

In certain cases, such as $\partial\Omega = S^3$, there are natural sub-Riemannian structures on $\partial\Omega$, such as, in this case, structures invariant with respect to the Lie group structure of S^3.

1.2.3 Heisenberg groups

Heisenberg groups are the simplest interesting example of sub-Riemannian spaces, and (along with other Carnot groups) provide an infinitesimal model of any sub-Riemannian space. The n^{th} Heisenberg group is defined (as a set and a group) as:

$$\mathbb{H}^n = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$$
$$(x,y,t) + (x',y',t') = \left(x + x', y + y', t + t' + \frac{1}{2}(x \cdot y' - y \cdot x')\right)$$

The sub-Riemannian structures is given by declaring the following left-invariant vector fields orthonormal (for $i = 1, \ldots, n$) and calling their (pointwise) span $\mathbb{H}$:

$$X_i = \frac{\partial}{\partial x_i} - \frac{1}{2} y_i \frac{\partial}{\partial t}, \quad Y_i = \frac{\partial}{\partial y_i} + \frac{1}{2} x_i \frac{\partial}{\partial t}$$

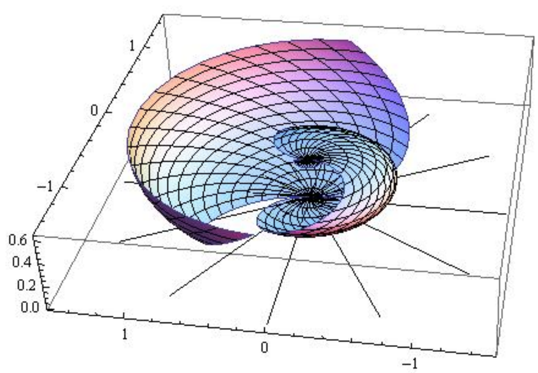

Figure 2: Some Geodesics from the origin in $\mathbb{H}^1$. The surfaces shown are ruled by geodesics but not horizontal (except for $t = 0$).

1. Since left-multiplication is linear, it is clear that the vector fields are left-invariant.
2. $[X_i, Y_i] = \frac{\partial}{\partial t}$, and other brackets give 0. The distribution is bracket-generating.
3. The dilation $\delta_t(x,y,t) = (tx, ty, t^2 t)$ dilates the metric d_{CC} by a factor of t.
4. As in the Grushin plane, the geodesics display spiraling behavior in the missing direction:
5. Again, while the rank of H is $2n$, the space of geodesics through a point is $2n+1$-dimensional.
6. As it turns out, the horizontal distribution may be extended to the one-point compactification of $\mathbb{H}^n$, and the resulting space is isomorphic to the structure on S^{2n-1} described above. In fact, the sub-Riemannian structure was first defined in order to prove the Mostow rigidity theorem for complex hyperbolic space, and is a limit (along horospheres) of the metric on complex hyperbolic space.

7. Lastly, $\mathbb{H}^n$ has a variety of applications. In particular, it comes up in the study of quantum mechanics. There, one models a system of n particles by taking the directions X_i as position values, and Y_i as momentum variables. The Lie bracket provides the link between the corresponding directions and momenta. The Heisenberg group then becomes (after a unitary representation into the automorphism group of a Hilbert space) the basic group of similarities of the model.

2 Tangent Approximation

2.1 How Standard Tangent Approximation Breaks Down

Consider a Riemannian manifold (M, g). The smoothness of g is equivalent to saying that the exponential map at any point $p \in M$ approximates the metric on M by a Euclidean metric on T_pM, and that this approximation holds to any order. In particular, the metric is locally invariant under standard Euclidean dilations. Even the group structure on the vector space can actually be deduced by dilating the metric at nearby points.

For a transformation $F : M \to N$, $p \in M$, and $q = \exp_p(\vec{v})$ close to p, we also have

$$\frac{d(F(p), F(q))}{d(p,q)} \approx \frac{|dF(p)\vec{v}|}{|\vec{v}|}$$

Both of these ideas fail in the sub-Riemannian case.:

1. The exponential map is not defined in non-horizontal directions.
2. For horizontal vectors, the exponential map cannot be defined uniquely. For example, in the Grushin plane, multiple curves leave the origin with original velocity in the x direction.
3. Consider the dilation $\delta_t(x, y) = (tx, t^2y)$ on the Grushin plane. The transformation dilates the metric by a factor t, but the action on the tangent space of the origin is given by the matrix $\begin{bmatrix} t & 0 \\ 0 & t^2 \end{bmatrix}$. This doesn't provide a satisfying estimate of the metric distortion.

2.2 Summary of the Heisenberg Manifold Case

We now focus on repairing the situation in the specific case of Heisenberg Manifolds (defined below). We follow Ponge 2000 and give his numbering in parentheses.

The main steps in the construction are as follows:

1. Give T_pM a graded Lie algebra structure $\mathfrak{g}_pM$ such that the first level is H_p and generates $\mathfrak{g}_pM$. The idea is that this space will model the directions at p. We unite these into a Lie algebra bundle $\mathfrak{g}M$. We use a theoretical approach that allows us to classify $\mathfrak{g}_pM$ in terms of a bilinear form.
2. We formally exponentiate $\mathfrak{g}_pM$ to get a Lie group G_pM that will model the metric properties of M at p. These give the Lie group bundle GM.

3. We repeat the previous constructions on the level of vector fields on TM and find "Heisenberg coordinates" that provide a logarithmic map from the neighborhood of a point to G_pM.

4. Ponge then shows that a Heisenberg manifold diffeomorphism is $F : M \to N$ is locally approximated by the induced map on the Lie group bundle.

We will also mention the more general construction of Bellaïche which shows that G_pM is correctly approximates the metric on M.

Definition 7. A Heisenberg manifold is a manifold M^{d+1} along with a horizontal distribution H and vector field $X_0, \dots X_d$ such that

$$TM = \langle X_0, \dots, X_d \rangle, \quad H = \langle X_1, \dots, X_d \rangle$$

Note that H must then have constant rank d, and that we are not fixing a metric on M.

Examples of Heisenberg manifolds include foliations, contact manifolds, CR manifolds, and Heisenberg groups.

(Note: 50-minute mark. We took a break here.)

2.3 Reorganizing the Tangent Bundle

Lemma 8. *(2.3) There is a two-form, called the Levi form, defined as:*

$$\begin{aligned} \mathfrak{L} :& H \times H \to TM/H \\ & (X, Y) \mapsto [X, Y] \ mod \ H \end{aligned}$$

Note that the two-form takes values in the line bundle TM/H, and that $H \times H$ is the bundle product.

The key point here is that $\mathfrak{L}(X, Y)_m$ depends only on the values X_m and Y_m.

We now use the Levi form to split the tangent bundle into the horizontal bundle and a complementary line bundle:

Definition 9. $\mathfrak{g}M = (TM/H) \oplus H$ as bundles. Let χ, χ' be sections of TM/H and X, X' be sections of H. We define the following structure on $\mathfrak{g}M$:

1. A grading on $\mathfrak{g}M$: H is the first level of the grading and TM/H the second.

2. A Lie bracket that agrees with the grading: $[\chi + X, \chi' + X']_m = \mathfrak{L}(X, X')$.

3. A dilation map that commutes with the Lie bracket: $\delta_t(\chi + X) = t^2\chi + tX$.

It is clear that $\mathfrak{g}M$ is then a 2-step nilpotent Lie algebra bundle with TM/H as the center (Proposition 2.6). Note, however, that the algebraic structure of $\mathfrak{g}M$ may vary from point to point.

In the case of nilpotent Lie algebras, we may use the Campbell-Hausdorff formula to define the associaed simply connected Lie group. Recall that for the 2-step case, the Campbell-Hausdorff formula states:

$$(\exp X)(\exp Y) = \exp\left(X + Y + \frac{1}{2}[X,Y]\right)$$

Furthermore, in the case of a 2-step nilotent Lie algebra $\mathfrak{g}$, we may take the associated simply connected Lie group G to be $\mathfrak{g}$ itself on the level of sets, with the exponential map equal to the identity, and define the group structure using the Campbell- Hausdorff formula.

Definition 10. By the above argument, we may view the Lie algebra bundle $\mathfrak{g}M$ as a Lie group bundle GM.

We can classify the groups G that appear in this fashion using the Levi form:

Proposition 11. *(Proposition 2.8) For $m \in M$, $\mathfrak{L}$ has rank $2n$ if and only if $G_m M \cong \mathbb{H}^n \times \mathbb{R}^{d-2n}$. Likewise, $\mathfrak{L}$ has constant rank $2n$ if and only if GM is an $\mathbb{H}^n \times \mathbb{R}^{d-2n-1}$-bundle over M.*

2.4 Construction of the Logarithmic Map

As far as I understand, there is no analogue of the exponential map from $G_m M$ to M. However, one can construct a local "logarithmic" map from a neighborhood of m to $G_m M$.

Definition 12. A coordinate chart ϕ satisfying the following conditions gives priveleged coordinates on M at m:

$$\phi_m : O \to \mathbb{R}^{d+1} \text{ for some open set } O \subset M$$
$$\phi_m(m) = 0$$
$$(\phi_m)_*(X_i(0)) = \frac{\partial}{\partial x_i} \text{ for } i = 0, \ldots, d+1$$

We will now focus on a point $m \in M$ and work in priveleged coordinates $(x_0, \ldots, x_d)$ at that point.

Definition 13. Recall that $T_m M = \langle X_0, \ldots, X_d \rangle_m$, and $X_1, \ldots, X_d$ generate the horizontal subspace. As before, we define a dilation that acts differently on x_0:

$$\delta_t(x_0, \ldots, x_d) = (t^2 x_0, t x_1, \ldots, t x_d).$$

We also define an operator δ_t^* that acts on functions f by

$$\delta_t^*(f) = f \circ \delta_t.$$

The action extends to vector fields X by conjugation:

$$(\delta_t^*(X))(f) = \delta_t^* X(\delta_{-t}^*(f)) = \delta_t^* X(f \circ \delta_{-t}) = X(f) \cdot (X(\delta_{-t}) \circ \delta_t)).$$

That is,

$$\delta_t^*(X) = (X(\delta_{-t}) \circ \delta_t) X$$

Definition 14. The goal now is to "straighten out" the vector fields $X_0, \dots, X_d$ at the origin by making them homogeneous. Define:

$$X_0^{(m)} := \lim_{t\to 0} t^2 \delta_t^*(X_0) = \frac{\partial}{\partial x_0}$$
$$X_j^{(m)} := \lim_{t\to 0} t\delta_t^*(X_j) = \frac{\partial}{\partial x_j} + \sum_{k=1}^{d} b_{jk} x_k \frac{\partial}{\partial x_0}, \quad j = 1, \dots, d,$$

where b_{jk} are some constants and x_k is the k^{th} coordinate function. Note that $X_j^{(m)}$ is homogeneous of degree 2 for $j = 0$ and degree 1 otherwise.

Lemma 15. *Let $\mathfrak{g}^{(m)}$ be the linear span of $\{X_0^{(m)}, \dots, X_d^{(m)}\}$ in the space of vector fields on $\mathbb{R}^{d+1}$. Then $\mathfrak{g}^{(m)}$ is a 2-step nilpotent Lie algebra.*

Lemma 16. *The coordinate space $\mathbb{R}^{d+1}$ can be given a Lie group structure $G^{(m)}$ so that $\mathfrak{g}^{(m)}$ is left-invariant. Thus, the coordinate chart ϕ has its image in a Lie group whose structure is determined by the algebraic properties of the vector fields X_i at m.*

Lemma 17. *There is an explicit diffeomorphism $\Psi_m : G^{(m)} \to G_m M$.*

Definition 18. The compositon $\Phi_m = \Psi_m \circ \phi_m : O \to G_m M$ is called a <u>Heisenberg coordinate map</u>.

2.5 Diffeomorphism Approximation

Let $F : (M, H) \to (N, H')$ be a Heisenberg diffeomorphism (that is, $df(H) = H'$).

Definition 19. Recall that we have $\mathfrak{g}M = TM/H \oplus H$ and let $\chi \in TM/H, X \in H$. Define

$$\begin{aligned} \delta F : &\mathfrak{g}M \to \mathfrak{g}N \\ &\chi + X \mapsto \overline{dF(\chi)} + dF(X) \end{aligned}$$

where $\overline{dF(\chi)}$ is the equivalence class of $dF(\chi)$ in TM/H. The action of δF on GM is the same.

Proposition 20. *(Proposition 2.3)The map δF is the linear approximation of F:*

$$\lim_{t\to 0} \frac{F(\delta_t x) - \delta F(x)}{t} = 0$$

(Note that $\delta \neq \delta_t$.) The convergence is locally uniform with respect ot both x and m.

2.6 Metric Approximation

Definition 21. The <u>Cygan norm</u> on $\mathbb{R}^{d+1}$ is $\|x\|_C = (x_0^2 + (x_1^2 + \dots + x_d^2)^2)^{\frac{1}{4}}$.

Bellaïche showed that $(G_m M, \|\cdot\|_C)$ approximates the metric on M near m. To state this rigorously, we need a few definitions.

Note: this holds for any sub-Riemannian structure on (M, H) just as a Riemannian metric is locally approximated by a Euclidean one.

Definition 22. 1. Let $A, B \subset X$ be two subsets of a metric space. The Hausdorff distance between them is

$$d_H(A, B) = \inf\{d \geq 0 \mid N_d(A) \supset B, N_d(B) \supset A\}$$

where N_d is a neighborhood of radius d.

2. Let X, Y be metric spaces.

$d_H(X, Y) = \inf\{d_H(i_1(X), i_2(Y)) \mid i_1, i_2$ are isometric embeddings into a metric space $Z\}$

3. Let $(X_n, x_n)_{n\in\mathbb{N}}, (Y, y)$ be pointed metric spaces. We say

$$\lim_{n\to\infty}(X_n, x_n) = (Y, y)$$

if for all $R > 0$ we have

$$\lim_{n\to\infty} d_H(B(x_n, R), B(y, R)) = 0$$

for all metric balls $B(x_n, R)$ of radius R centered at x_n. The idea is that the metric balls of X_n look more and more like the metric balls of Y.

4. For a metric space X and $x \in X$, the metric tangent space of X at x is

$$(T_x X, 0) = \lim_{t\to\infty}(tX, p),$$

provided it exists. Here, tX refers to the space X whose distances have been dilated by t. Note that $T_x X$ is dilation-invariant.

Theorem 23. *(Bellaïche) $(G_m M, \|\cdot\|_C)$ is the tangent space of M at m for any sub-Riemannian metric g on (M, H).*

Note: the theorem is in fact more general, defining $G_m M$ for a general bracket-generating sub-Riemannian structure and showing it to be the metric tangent space at m.

3 Final Remarks

3.1 Conclusion

Reorganizing the tangent bundle of M has given a way to approximate the metric structure of sub-Riemannian manifolds as the Cygan metric on nilpotent Lie groups. Redefining the action on the tangent space allows us to study mappings between sub-Riemannian manifolds. However, there has been little work in this direction, and the standard differential geometry methods have not been adjusted to this setting. For example, conditions and statements for implicit and inverse function theorems, identification of maxima/minima, etc, have not been studied.

3.2 References

1. Raphaël Ponge, "The Tangent Groupoid of a Heisenberg Manifold", http://arxiv.org/abs/math/0404174

2. André Bellaïche, "Tangent Space in sub-Riemannian Geometry", in Sub-Riemannian Geometry by Bellaïche et al., Birkhüser (note: a significantly shorter version of this article was also published in a journal).