

## Clifford Torus Calculation

### Standard Construction

The standard Clifford torus in CHKG is defined by the vectors  $w_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ ,  $w_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ .

The Cayley transformation  $C$  is  $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{bmatrix}$ .

Under the Cayley transformation  $w_1$  becomes  $Cw_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ ,  $J_2$ -orthogonal to  $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix}$ . These two vectors define the chain seen as the  $Z$ -axis in Heisenberg space.

Likewise,  $Cw_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ ,  $J_2$ -orthogonal to  $\begin{bmatrix} -1 \\ \sqrt{2} \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} -1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$ , defining the unit circle in the  $XY$  plane in Heisenberg space.

The torus itself is given in the  $J_1$  model by  $|z_1|^2 = |z_2|^2 = \frac{1}{\sqrt{2}}$ , which can be parametrized and pushed forward by the Cayley transform and then map to Heisenberg space to produce a Heisenberg depiction.

### Modificaton

I'd like to adjust the radius of the second chain and see the resulting changes in the torus.

Consider the Heisenberg points  $\begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} -r \\ 0 \\ 0 \end{bmatrix}$ . These lift in  $J_2$  to  $\begin{bmatrix} -r \\ \sqrt{2}r \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} -r \\ -\sqrt{2}r \\ 1 \end{bmatrix}$ ,

$J_2$ -orthogonal to  $\begin{bmatrix} r \\ 0 \\ 1 \end{bmatrix}$ . Under the Cayley transformation (removing the  $\frac{1}{\sqrt{2}}$ ) it becomes  $\begin{bmatrix} r+1 \\ 0 \\ r-1 \end{bmatrix}$ .

Normalizing to  $J_1$  norm-squared  $4r$ , the new defining vectors are then  $w_1 = \begin{bmatrix} 0 \\ 4r \\ 0 \end{bmatrix}$  and  $w_2 = \begin{bmatrix} r+1 \\ 0 \\ r-1 \end{bmatrix}$ . The Clifford cone is then given by the condition (for  $w = \begin{bmatrix} z_1 \\ z_2 \\ 1 \end{bmatrix}$ )

$$|\langle w, w_1 \rangle| = 4r |z_2| = |(r+1)z_1 + (r-1)| |\langle w, w_2 \rangle| \quad (1)$$

The Clifford torus has the additional condition  $|z_1|^2 + |z_2|^2 = 1$ . This gives

$$|z_1|^2 + \frac{1}{16r^2} |(r+1)z_1 + (r-1)|^2 = 1 \quad (2)$$

Taking  $z_1 = x + iy$  and expanding,

$$16r^2(x^2 + y^2) + (x(r+1) + r-1)^2 + (r+1)^2y^2 = 16r^2 \quad (3)$$

$$(17r^2 + 2r + 1)y^2 + (17r^2 + 2r + 1)x^2 + 2(r^2 - 1)x + r^2 - 2r + 1 = 16r^2 \quad (4)$$

Completing the square,

$$(17r^2 + 2r + 1)y^2 + (17r^2 + 2r + 1) \left( x + \frac{r^2 - 1}{\sqrt{17r^2 + 2r + 1}} \right)^2 - \frac{(r^2 - 1)^2}{17r^2 + 2r + 1} + r^2 - 2r + 1 = 16r^2 \quad (5)$$

$$y^2 + \left( x + \frac{r^2 - 1}{\sqrt{17r^2 + 2r + 1}} \right)^2 = \frac{64r^3(1 + 4r)}{(17r^2 + 2r + 1)^2} \quad (6)$$

This is parametrized as:

$$\left[ \begin{array}{c} \left( \frac{\sqrt{64r^3(1+4r)}}{17r^2+2r+1} \cos(\theta) - \frac{r^2-1}{\sqrt{17r^2+2r+1}} \right) + \frac{\sqrt{64r^3(1+4r)}}{17r^2+2r+1} i \sin(\theta) \\ \sqrt{1 - |z_1|^2} e^{i\phi} \\ 1 \end{array} \right] \quad (7)$$

and yields a condition for  $|z_2|$  that I leave to Mathematica to figure out. Again, pushing to  $J_2$  and to Heisenberg space produces pictures.