Exam 1, Version B Name: Anton

Some problem-solving hints:

1. Don't panic.
2. If you're stuck, at least try *something*.
3. If you can't do something, don't.
4. If things gets weird, there's probably a mistake.
5. If you can't solve a problem, solve an easier problem first.
6. When in doubt, write it out.
7. Remember: $(a+b)^2 \neq a^2 + b^2$.

Laws of limits:

Suppose $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, and c is a constant. Then:

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$
4. $\lim_{x\to a}[f(x)g(x)] = (\lim_{x\to a} f(x)) \cdot (\lim_{x\to a} g(x))$
5. $\lim_{x\to a}[f(x)/g(x)] = (\lim_{x\to a} f(x)) \,/\, (\lim_{x\to a} g(x))$ if $\lim_{x\to a} g(x) \neq 0$
6. $\lim_{x\to a} c = c$
7. $\lim_{x\to a} x = a$

Problem 1. *(6 points) Simplify the following:*

a) $\frac{(4^3 \cdot 5^{-2})^2}{2^6 \cdot 7}$ $= \frac{4^6 \cdot 5^{-4}}{2^6 \cdot 7} = \frac{(2^2)^6 \cdot 5^{-4}}{2^6 \cdot 7}$

$= \frac{2^{12} \cdot 5^{-4}}{2^6 \cdot 7} = \frac{2^6}{5^4 \cdot 7} = \frac{64}{625 \cdot 7} = \boxed{\frac{64}{4385}}$

b) $\sqrt{x^2}$

$|x|$

c) $e^{2 \ln 5}$ $= e^{\ln 5^2} = 5^2 = 25$

Problem 2. *(4 points) What is the domain of* $\sqrt{|x|}$*?*

$\mathbb{R}$

Problem 3. *(10 points) Compute* $\lim_{x\to 2} \frac{x^2}{x+3}$ *using the Limit Laws (no short-cuts!). Use only one limit law in each step and tell me which law you are using.*

$$\lim_{x\to 2} \frac{x^2}{x+3} \overset{5}{=} \frac{\lim_{x\to 2} x^2}{\lim_{x\to 2}(x+3)} \overset{4}{=} \frac{\left(\lim_{x\to 2} x\right)\left(\lim_{x\to 2} x\right)}{\lim_{x\to 2}(x+3)}$$

$$\overset{7}{=} \frac{2\cdot 2}{\lim_{x\to 2}(x+3)} = \frac{4}{\lim_{x\to 2}(x+3)} \overset{1}{=} \frac{4}{\lim_{x\to 2} x + \lim_{x\to 2} 3}$$

$$\overset{6}{=} \frac{4}{\lim_{x\to 2} x + 3} \overset{7}{=} \frac{4}{2+3} = \boxed{5}$$

Problem 4. *(10 points) Suppose we have the following income tax system: for the first \$10,000 there is no tax. Any income past \$10,000 is taxed at 10%.*

a) Write the amount of tax paid as a function $f(x)$ of the income.

$$f(x) = \begin{cases} 0 & x \leq 10{,}000 \\ 0.1\,(x - 10{,}000), & x \geq 10{,}000 \end{cases}$$

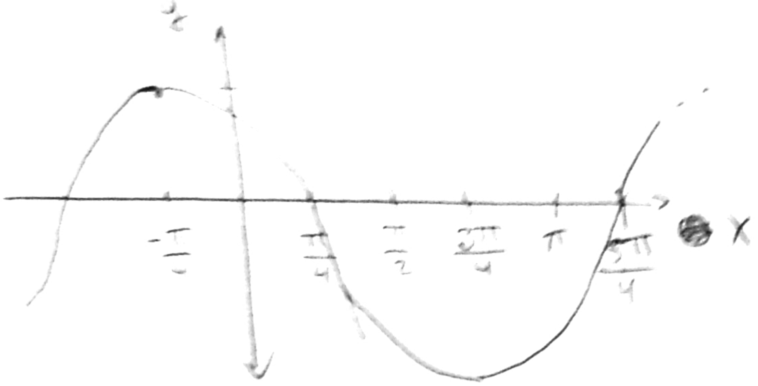
b) What is $f'(x)$?

$$f'(x) = \begin{cases} 0 & x < 10{,}000 \\ .1 & x > 10{,}000 \end{cases}$$

c) What is the significance of $f'(x)$ in this situation?

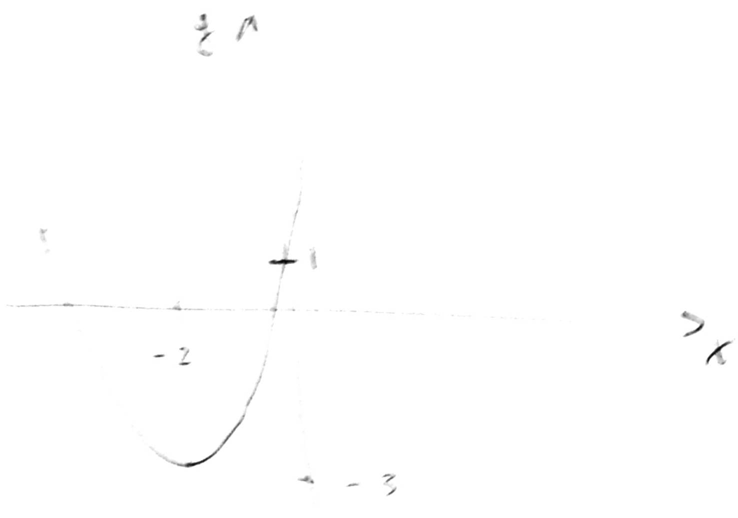
tax paid per dollar income, at the given income level.

Problem 5. *(5 points) Graph* $y = \cos(x + \pi/4)$.



Problem 6. *(5 points) Graph* $y = x^2 + 4x + 1$ *by completing the square.*

$$y = (x+2)^2 - 3$$



Problem 8. *(15 points) Let* $f(x) = \frac{x^2 - x - 2}{x-2} = \frac{(x+1)(x-2)}{x-2}$

a) *Compute* $\lim_{x\to 2} f(x)$.

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{(x+1)(x-2)}{x-2} = \lim_{x\to 2} x+1 = \boxed{3}$$

b) *What does it mean for a function to be continous at a point* a?

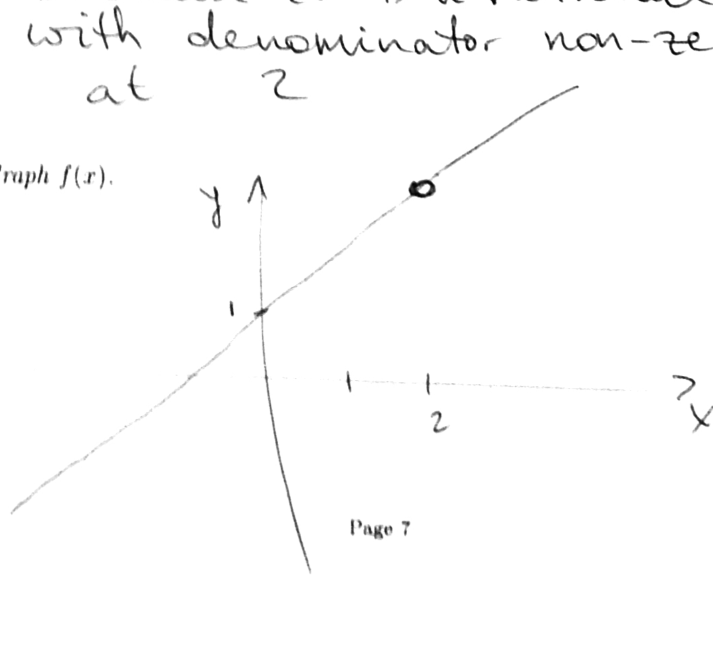
$$\lim_{x\to a} f(x) = f(a)$$

c) *At what points is* f *continuous?*

$(-\infty, 2) \cup (2, \infty)$

because it is a rational function with denominator non-zero except at 2

d) *Graph* $f(x)$.



Problem 7. *(15 points) Let* $f(x) = \dfrac{(x-1)^2}{x^2}$.

a) Compute all vertical or horizontal asymptotes for f.

$\underline{x = 0}$

$$\lim_{x \to 0} \frac{(x-1)^2}{x^2} \cancel{= \lim_{x\to 0} \frac{x^2 - 2x + 1}{x^2}}$$

vertical

$$= \lim_{x \to 0} \left(\frac{x-1}{x}\right)^2 \cancel{= \lim_{x\to 0} (}$$

$$= \left(\lim_{x \to 0} \frac{x-1}{x}\right)^2 = \left(\lim_{x \to 0} 1 - 1/x\right)^2 = (\pm\infty)^2 = \infty$$

$\underline{x \to \infty}$

$$\lim_{x \to \infty} \frac{(x-1)^2}{x^2} \cancel{= \lim_{x\to\infty} \frac{x^2 - 2x + 1}{x^2}} \cancel{= \lim_{x\to\infty} 1 - \frac{2}{x} + \frac{1}{x^2}}$$

horizontal

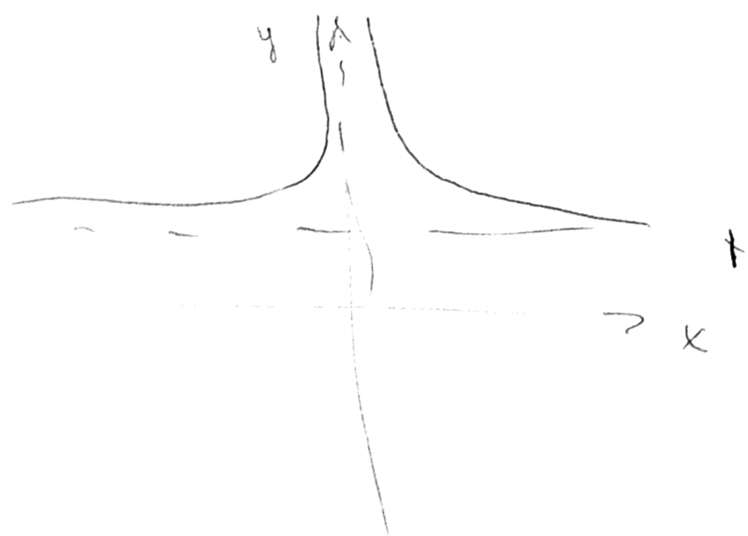
$$= \lim_{x \to \infty} \left(\frac{x-1}{x}\right)^2$$

$$= \left(\lim_{x \to \infty} 1 - \frac{1}{x}\right)^2 = 1^2 = 1$$

horizontal $\underline{x \to -\infty}$

$$\lim_{x \to -\infty} \frac{(x-1)^2}{x^2} = \lim_{x \to -\infty} \left(\frac{x-1}{x}\right)^2 = \left(\lim_{x \to -\infty} \frac{x-1}{x}\right)^2 = 1^2 = \boxed{1}$$

b) Graph $f(x)$ *using the previous answer.*



Problem 9. *(15 points) Let* $f(x) = \begin{cases} x^2\sin(1/x) + x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

a) Is $f(x)$ continuous at 0? Prove that you are correct.

Yes, by squeeze theorem:

$$\underbrace{-x^2 - |x|}_{} \leq f(x) < \underbrace{x^2 + |x|}_{}$$

$\lim_{x \to 0}$ is 0

So by squeeze theorem, $\lim_{x\to 0} f(x) = 0$, which is also $f(0)$.

Everywhere else, f is a composition of polynomials, rational, and trig functions, so we know it's continuous there.

b) Does $f'(0)$ exist? If it does not exist, explain why not. If it does exist, what is its value?

Yes:

$$f'(0) = \lim_{h \to 0} \frac{h^2 \sin(1/h) + h - f(0)}{h}$$

$$= \lim_{h\to 0} \underbrace{h \sin(1/h) + 1}_{} = 1$$

again, by squeeze theorem, since

$$-|h| + 1 \leq h \sin(1/h) + 1 < |h| + 1$$

$\to \cancel{0}\, 1$ $\qquad \to 1$

Problem 10. *(10 points) Use the limit definition of derivative to compute* $\frac{d}{dt}(5t - 9t^2)$. *Do NOT use any shortcuts.*

$$\lim_{h \to 0} \frac{5(t+h) - 9(t+h)^2 - 5t + 9t^2}{h}$$

$$= \lim_{h \to 0} \frac{5t + 5h - 9t^2 - 9h^2 - 18h - 5t + 9t^2}{h}$$

$$= \lim_{h \to 0} \frac{5h - 9h^2 - 18h}{h} = 5 - 18 = -13$$

Problem 11. *(10 points) Does there exist a number whose sine is equal to its square root? Your explanation should be very precise and clear (use full sentences).*

Yes. $\sin a = \sqrt{a}$?

Let $f(a) = \sin a - \sqrt{a}$, continuous everywhere.

~~For~~ $f(0) = 0$

Oh, $\sin(0) = \sqrt{0}$!

Ok, $a = 0$ works so don't have to use IVT.