Math 675 Potential Final Exam Problems

The exam will consist of 20 problems, separated into definitions, examples, and proofs. Here are some potential questions.

The list of questions may be modified before the actual exam is compiled. You should think about what additional problems might be added.

Definitions

1. Metric
2. Closure
3. Vector space
4. Contact point
5. Cauchy
6. Complete
7. Sequentially compact
8. Compact (set)
9. Separable
10. Open
11. Closed
12. Continuous
13. Homeomorphism
14. Isometry
15. Completion
16. Contraction
17. Linearly independent
18. Basis
19. Span
20. Linear functional
21. Null space
22. Hyperplane
23. Codimension
24. Convex
25. Convex body
26. Convex functional
27. Quotient space
28. Minkowski functional
29. Subspace
30. d_p
31. $C_p[a,b]$
32. ℓ_p

33. Norm
34. Inner product
35. Bessel's inequality
36. Parseval's identity
37. Fourier coefficient
38. Hilbert space
39. Banach space
40. Hölder's inequality
41. Cauchy's inequality
42. Orthogonal complement
43. Riesz-Fisher Theorem
44. Continuous linear functional
45. Dual space
46. Dual operator
47. Spectrum
48. Adjoint
49. Eigenvalue
50. Eigenvector
51. Completely continuous
52. Compact operator
53. Self-adjoint
54. Weierstrass approximation theorem

Proofs

1. For $a < b$ in $\mathbb{R}$, the closure of (a, b) is $[a, b]$.
2. In $\mathbb{R}^n$, the metric d_∞ is the limit of the functions d_p as $p \to \infty$.
3. The spaces $C_\infty[a, b]$ and $C_\infty[c, d]$ are isometric for any $a < b$ and $c < d$.
4. A point $x \in X$ is a limit point of A if and only if for every $\epsilon > 0$ the ball $B_\epsilon(x)$ contains a point of A not equal to x.
5. The closure of A equals the union of the set of isolated points A with the set of limit points of A.
6. What is the closure of $\mathbb{Q} \subset \mathbb{R}$? Prove that you are correct.
7. The following are equivalent for a function $f : X \to Y$ between two metric spaces:
   (i) f is continuous,
   (ii) $f^{-1}(A)$ is an open set for every open set $A \subset Y$,
   (iii) $f^{-1}(B)$ is a closed set for every closed set $B \subset Y$.
8. The set of invertible 2-by-2 matrices is open. (Interpret the space of all 2-by-2 matrices as $\mathbb{R}^4$ with the usual topology.) You may use your favorite definition of continuity.

9. The set of determinant-one 2-by-2 matrices is closed. You may use your favorite definition of continuity.
10. Let A be a linearly independent set in a metric vector space V. Assuming that V is finite-dimensional, prove that the set $\mathbb{R}A = \{\lambda a : \lambda \in \mathbb{R}, a \in A\}$ is closed.
11. Suppose two metric spaces X and Y are bi-Lipschitz equivalent. Prove that that X is complete if and only if Y is complete.
12. The image of a compact set under a continuous function is compact.
13. Let $f : X \to X$ be a contraction, and $Y \supset X$ a completion of X. Prove that f extends to a contraction of Y. Use this to conclude that there is no homeomorphism of $(0, 1)$ that is also a contraction.
14. The contraction theorem requires a mapping that is L-Lipschitz for $L < 1$. Prove that the result is sharp. That is, allowing $L = 1$ in the theorem would make it false.
15. A closed subset of a compact space is compact.
16. Suppose f, f_1, f_2 are linear functionals such that $f(x) = 0$ if $f_1(x) = 0$ and $f_2(x) = 0$. Prove that there are constants a_1 and a_2 such that $f = a_1 f_1 + a_2 f_2$.
17. If X is compact and $f : X \to \mathbb{R}$ is continuous, then f attains a maximum value.
18. If $A, B \subset X$ are compact subsets of a metric space X, then $A \cup B$ is compact.
19. Suppose $A, B \subset V$ are separated by a linear functional f. If A is a convex body, prove that there are points $a \in A$ and $b \in B$ such that $f(a) \neq f(b)$.
20. Let $\|\cdot\|$ be the norm in $\mathbb{R}^2$ for which the unit circle is a regular hexagon with side length 1. Prove that $\|\cdot\|$ is not induced by an inner product. (Hint: the parallelogram law states $2(a^2 + b^2) = (a - b)^2 + (a + b)^2$).
21. Given a Banach space V, let $\{B_n\}$ be a nested sequence of closed spheres in V. Prove that $\cap_n B_n$ is nonempty. (Note that the radius is not assumed to go to zero, and the centers are not assumed to be the same.)
22. Let $\{e_i\}$ be an orthonormal basis for a Hilbert space H. Take $x_n = e_{2n}$ and $y_n = e_{2n} + \frac{e_{2n+1}}{n+1}$, and set $M = \text{span}(\{x_n\})$, $N = \text{span}(\{y_n\})$. Show that $M + N$ is not closed, even though M and N are both closed.
23. Prove that the functional $F(f) = \int_a^b f(t) \cos(t) dt$ is a continuous functional on $C_\infty[a, b]$. What could we replace $\cos(t)$ with?

24. Prove that both of the following is a subspace of ℓ^2: the set of all (x_i) such that $x_1 = x_2$.
25. Prove that both of the following is a subspace of ℓ^2: the set of all (x_i) such that $x_k = 0$ for all even k.
26. Let V be a normed real vector space and $F : V \to \mathbb{R}$ a linear functional. Prove that F is continuous if and only if $N(F)$ is closed. (Hint: The forward direction requires no work at all; for the backward direction, assume F is not bounded and show that $N(F)$ is not closed by perturbing a non-zero point.)
27. Prove that following functional is continuous on $C_\infty[0,1]$ and compute its norm: $f(x) = ax(0) + bx(1)$.
28. Prove that following functional is continuous on $C_\infty[0,1]$ and compute its norm: $g(x) = \int_0^{1/2} x(t)dt - \int_{1/2}^1 x(t)dt$.
29. Prove that if $p < q < \infty$ and f is a linear functional on $C_p[0,1]$ then it is also continuous on $C_q[0,1]$.
30. Let V_0 be a normed real vector space and V its completion. Prove that V_0^* and V^* are isomorphic Banach spaces.
31. Recall that a sequence v_i in a normed space V converges weakly to v if $f(v_i)$ converges to $f(v)$ for each linear functional $f \in V^*$. Prove that the standard basis vectors e_i in ℓ_2 weakly converge to the zero vector.
32. Prove that an infinite-dimensional Banach space has uncountable algebraic dimension.
33. Prove that every element of $C_2[a,b]$ is of the form $\sum_1^\infty a_i p_i(x)$ for $a_i \in \mathbb{R}$ and p_i a polynomial of degree i.
34. A linear functional is continuous if and only if it is bounded.
35. Let $V \subset C_\infty[a,b]$ consist of all continuously differentiable functions. Prove that the differentiation operator $D : V \to C_\infty[a,b]$ is not continuous.

Examples

1. A bounded set that is not compact.
2. A metric vector space that is not separable.
3. A linearly independent set A in a metric vector space V such that the set $\mathbb{R}A = \{\lambda a : \lambda \in \mathbb{R}, a \in A\}$ is not closed.
4. A union of closed sets that is not closed.
5. An intersection of open sets that is not open.
6. Homeomorphic metric spaces X and Y such that X is complete but Y isn't.
7. A compact set C and a continuous function f such that $f^{-1}(C)$ is not compact.

8. A contraction that does not have a fixed point.
9. Let V be a non-trivial vector space. Give an example of a non-trivial functional on V^*.
10. A convex set that is not a convex body.
11. Two disjoint sets in $\mathbb{R}^2$ that cannot be separated by a linear functional.
12. A Cauchy sequence in $C_2[a,b]$ that does not converge to a function in $C_2[a,b]$.
13. An explicit linear functional on $\mathbb{R}^2$ whose null space is spanned by the vector $(1,2)$.
14. An explicit convex functional on $\mathbb{R}^2$.
15. A function in the completion of $C_1[0,1]$ that is not in the completion of $C_\infty[0,1]$.
16. A function in the completion of $C_1[0,1]$ but that is not in the completion of $C_2[0,1]$.
17. A linear function that is continuous on $C_\infty[0,1]$ but not on $C_1[0,1]$.
18. A continuous operator that is not compact.
19. A compact operator that has no eigenvalues.
20. A continuous operator that has no eigenvalues.
21. An operator from a space to itself that is not self-adjoint.
22. An operator $A : \ell_2 \to \ell_2$ whose eigenvalues include 1 and 2.
23. A non-trivial operator $A : \ell_2 \to \ell_2$ whose eigenvalues are all smaller than $1/2$.
24. an inner product space V and an orthonormal system $\{e_i\}$ such that V contains no non-zero element orthogonal to every e_i, even though $\{e_i\}$ does not span V.
25. A nested sequence of closed non-empty sets C_i in $\mathbb{R}$ such that $\cap C_i = \emptyset$.
26. A nested sequence $\{E_n\}$ of nonempty closed bounded convex sets in a Banach space V (of your choice) such that $\cap_n E_n = \emptyset$.
27. A basis for $C_2[0,\pi]$.
28. An element of ℓ_∞ that cannot be written as $\sum_i^\infty a_i e_i$, where $a_i \in \mathbb{R}$ and e_i is the sequence $(0,\ldots,0,1,0,\ldots)$ where the 1 is in the i^{th} place.
29. An continuous operator A that is invertible such that A^{-1} is not continuous.
30. Recall that the *support* of a function is the closure of the set of points where it has non-zero values. Suppose $f \in C_2[0,1]$ is supported on $[0,1/2]$. Give an example of a non-trivial element of $f^\perp$.

31. Recall that V is reflexive if the canonical mapping from V to $(V^*)^*$ is an isomorphism. Give an example of a non-reflexive space.