Anosov representations in AdS geometry

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The point of this note is to summarize the work of T. Barbot and Q. Merigot on Anosov representations in anti de Sitter geometry, published as one joint paper [BM], but available on arXiv as two separate papers [Mer] and [Ba1]. Combined, the work of Barbot and Merigot shows that when the target group is $\mathrm{SO}(2,n)$, the notion of Anosov representation (Labourie [La]) coincides exactly with the notion of quasifuchsian: here the natural space on which $\mathrm{SO}(2,n)$ acts is the Einstein universe $\mathrm{Ein}^{n-1,1} = \mathrm{Ein}^n$, a conformal Lorentzian manifold which should be thought of as a Lorentzian analogue of the conformal Riemannian sphere. $\mathrm{SO}(2,n)$ also acts on the $n+1$ dimensional anti de Sitter space $\mathrm{AdS}^{n,1} = \mathrm{AdS}^{n+1}$, which is a Lorentzian model space of constant negative curvature, a Lorentzian analogue of hyperbolic space. The Einstein space $\mathrm{Ein}^{n-1,1}$ is the ideal boundary of $\mathrm{AdS}^{n,1}$, in the same way that the conformal sphere is the boundary of hyperbolic space.

Let Γ be a cocompact lattice is $\mathrm{SO}(n,1)$. Via the inclusion $\mathrm{SO}(n,1) \hookrightarrow \mathrm{SO}(n,2)$ we obtain a *Fuchsian* representation of Γ. This representation preserves a totally geodesic copy of $\mathbb{H}^n$ inside $\mathrm{AdS}^{n,1}$; it also preserves the boundary $\partial\mathbb{H}^n \subset \mathrm{Ein}^{n,1}$, which is a smooth space-like $(n-1)$-sphere in $\mathrm{Ein}^{n-1,1}$. A representation $\rho : \Gamma \to \mathrm{SO}(n,2)$ is *quasi-Fuchsian* if ρ preserves an embedded, *acausal*, $n-1$ sphere Λ (the limit set) in $\mathrm{Ein}^{n-1,1}$. When we say that a representation $\rho : \Gamma \to \mathrm{SO}(n,2)$ is *Anosov*, we will mean Anosov in the context of the action of $\mathrm{SO}(n,2)$ on the subspace $\mathcal{Y} \subset \mathrm{Ein}^n \times \mathrm{Ein}^n$ of pairs of points (v,w) that are *acausal*, meaning there exists a space-like path in (the universal cover) of Ein^n, connecting v to w. The key connection between the notion of Anosov and that of quasi-Fuchsian comes from AdS geometry: The limit set $\Lambda \subset \mathrm{Ein}$ of a quasi-Fuchsian representation ρ is the boundary of a domain in AdS whose quotient under ρ is a *globally hyperbolic Cauchy compact* (GHC) AdS manifold. The geodesic flow on the non-wandering space-like unit tangent bundle in this manifold turns out to be dynamically the same as the geodesic flow on $\Gamma\backslash T^1\mathbb{H}^n$; this allows us to construct the Anosov map from $T^1(\Gamma\backslash\mathbb{H}^n)$ to $\mathcal{Y}$.

Finally, we mention one newer result of Barbot [Ba2], which we will not discuss here: All deformations of the Fuchsian representation of Γ in $\mathrm{SO}(2,n)$ are quasi-Fuchsian. In other words the component of the Fuchsian representation consists entirely of Anosov representations. This is well-known due to Mess in dimension three.

Before starting, a disclaimer: I have left out many of the details (and perhaps even some important ones!). I only intend to give a vague idea of the arguments.

1 Definitions of the basic objects

For background on AdS geometry, see [BB] or my thesis (mostly dimension 3). For a nice introduction to the $2+1$ dimensional Einstein space, see [BCD+]

Consider the quadratic form

$$q_{2,n}(u,v,x_1,\dots,x_n) = -u^2 - v^2 + x_1^2 + \cdots + x_n^2$$

of signature $(n,2)$ on $\mathbb{R}^{n+2} = \mathbb{R}^{2,n}$. Let $\langle \cdot | \cdot \rangle$ denote the associated inner porduct. The anti de Sitter space $\mathrm{AdS}^{n+1} = \mathrm{AdS}^{n,1}$ is defined to be the hyperboloid $\{q_{2,n} = -1\}$ endowed with the metric coming from the restriction of $q_{2,n}$. This metric is *Lorentzian*, meaning that the inner product on each tangent space has signature $(1,n)$ (one negative eigenvalue and n positive eigenvalues).

The Einstein universe $\mathrm{Ein}^{n-1,1} = \mathrm{Ein}^n$ is the positive projectivization of the light cone with respect to $q_{2,n}$:

$$\mathrm{Ein}^{n-1,1} = \{q_{2,n}(u,v,\mathbf{x}) = 0 : \mathbf{x} \neq 0\}/\mathbb{R}^+.$$

It is topologically $\mathbb{S}^1 \times \mathbb{S}^n$; the diffeomorphism is given by

$$(u,v,\mathbf{x}) \mapsto \left(\frac{(u,v)}{\sqrt{u^2+v^2}}, \frac{\mathbf{x}}{\|\mathbf{x}\|} \right),$$

where $\|\cdot\|$ denotes the usual Euclidean norm.

The group $\mathrm{SO}(2,n)$ acts on both $\mathrm{AdS}^{n,1}$ and $\mathrm{Ein}^{n-1,1}$. In the case of AdS, its clear that this action is by isometries. Under the identification $\mathrm{Ein}^{n,1} = \mathbb{S}^1 \times \mathbb{S}^n$ above, the Lorentzian metric $-d\theta^2 + ds^2$ is preserved, *up to conformal factor*, by $\mathrm{SO}(2,n)$. This conformal metric is induced by $q_{2,n}$.

Let $\mathbb{S} : \mathbb{R}^{n+2} \setminus \{0\} \to \mathbb{S}^{n+1}$ denote the quotient by the positive reals $\mathbb{R}^+$. Then $\mathbb{S}$ is injective on both $\mathrm{AdS}^{n,1}$ and $\mathrm{Ein}^{n-1,1}$ and Merigot denotes the images $\mathbb{ADS}^{n,1} = \mathbb{S}(\mathrm{AdS}^{n,1})$ and $\partial\mathbb{ADS}^{n,1} = \mathbb{S}(\mathrm{Ein}^{n-1,1})$. The image of Einstein space is precisely the boundary of the image of AdS. This allows us to think of Ein as an ideal boundary of AdS in the same way that the sphere is the ideal boundary of $\mathbb{H}^n$. The (pushforward of the) metrics work well together: If p_n is a sequence of points in $\mathbb{ADS}$ approaching the point p_∞ in $\partial\mathbb{ADS}$, then the conformal class of the AdS metric at p_n converges to the (conformal class of the) Einstein metric at p_∞. One other important feature that can be seen directly in this model is that every space-like geodesic in $\mathbb{ADS}$ has two distinct end-points on $\partial\mathbb{ADS}$. Let $\mathcal{E}^1\mathrm{AdS}^{n,1}$ denote the unit space-like tangent bundle of AdS, consisting of all tangent vectors with norm-squared $+1$. Then, define the two maps $\ell_+, \ell_- : \mathcal{E}^1\mathrm{AdS}^{n,1} \to \mathrm{Ein}^{n-1,1}$ taking a tangent vector to its forward and backward endpoints (respectively) under the geodesic flow. These will be useful later. Henceforth, we will not distinguish between the two models $\mathbb{ADS}$ and AdS, and we will only use the notation AdS and Ein. Note that some (including me) prefer to take AdS to be the quotient by the antipodal map of the model defined here.

1.1 Causality

Let M be a manifold with Lorentzian metric g, and let $v \neq 0$ be a vector tangent to M at some point. If $g(v,v) < 0$, then v is called *time-like*, if $g(v,v) = 0$ then v is called light-like and if $g(v,v) > 0$, then v is called *space-like*. This terminology comes from physics (Einstein): paths with time-like tangents describe the motion of particles; a geodesic with light-like tangent describes a photon. A hyper-surface with space-like tangents is a copy of space in the space-time, you could think of it as the slice $t = 0$, although in general such a time function t might not extend to the entire space-time M. The notion of *globally hyperbolic* means roughly that there is a global time function t on M. Then, the level sets of t are space-like hypersurfaces.

In any tangent space, the light-like vectors form a cone which divides the time-like vectors into two components. We choose one component to be called *future* and the other to be called *past*. This choice can be made consistently over small patches of the manifold. The manifold is called *time-orientable* if future/past can be chosen consistently over the entire manifold. Even a time-orientable space-time may not have a well-defined notion of future/past for pairs of points. For example, both Ein and AdS contain closed time-like curves, so future/past only makes sense in the universal cover.

Since we wish to deal with rough sets (like limit sets), we need to extend the notion of time-like, space-like, light-like, to curves which are not smooth. Causality notions only make sense in the universal cover; when applied to a manifold which is not simply connected, lift everything to the universal cover before applying the definitions. A curve c is called (future) *causal* if for every $a > b$, $c(a)$ is in the future of $c(b)$, meaning that there exists a smooth curve from $c(a)$ to $c(b)$ with (future directed) time-like or light-like tangents. A set Λ is called *achronal* if for any two points of Λ there is no time-like curve separating those points. A set Λ is *acausal* if for any two points of Λ, there is no causal curve separating those points.

In our context, $\Lambda \subset \mathrm{Ein}^n$ will be the image of an $n-1$-sphere under some injective map. In this case:

Lemma 1. *Λ is acausal if and only if for all $p, q \in \Lambda$, the pair $(p, q) \in \mathcal{Y}$.*

This is not hard to prove. Let $\widetilde{\mathrm{AdS}}^{n,1}$ denote the universal cover, and let $\widetilde{\mathrm{Ein}}^{n-1,1}$ denote the boundary of $\widetilde{\mathrm{AdS}}$ ($n = 2$ is slightly annoying here since $\widetilde{\mathrm{Ein}}$ is not the universal cover of Ein so lets ignore this case here). Let $\widetilde{\Lambda}$ be a lift of Λ to $\widetilde{\mathrm{Ein}}$. If some pair $(p, q) \notin \mathcal{Y}$, then acausal fails: the span of p and q gives a light-like geodesic in Ein^n passing through p and q; this light-like geodesic generates $\pi_1\mathrm{Ein}$ so any lifts $\widetilde{p}, \widetilde{q}$ to $\widetilde{\mathrm{Ein}}$ are connected by the lift of the light-like geodesic. On the other hand, if all pairs of points $p, q \in \Lambda$ satisfy $(p, q) \in \mathcal{Y}$, then we show Λ is acausal as follows. Let $p, q \in \Lambda$. By the assumption $(p, q) \in \mathcal{Y}$, $\mathrm{span}\{p, q\}$ is a $(1, 1)$ plane which descends to a space-like geodesic in $\mathrm{AdS}^{n,1}$. This geodesic is contained in many totally geodesic hyperbolic planes. To find one, pick a positive vector v in $\mathrm{span}\{p, q\}^\perp$; then $\mathrm{span}(p, q, v)$ has signature $(1, 2)$ and descends to a hyperbolic plane H in $\mathrm{AdS}^{n,1}$. The boundary ∂H is a space-like curve in Ein^n passing through p and q. Now let α be the loop gotten by traveling from p to q along some path γ inside Λ and then traveling from q back to p along ∂H. Then a homotopy α_t from α to the constant loop can be constructed: α_t is the loop going from p to the point $\gamma(t)$ ($\gamma(0) = p$, $\gamma(1) = q$) along γ and going back to p along $\partial H_{\gamma(t)}$ where $H_\gamma(t)$ is a hyperbolic plane containing the geodesic from p to $\gamma(t)$ (chosen continuously in t; here we need that $(p, \gamma(t)) \in \mathcal{Y}$) for all t). Now lift Λ to $\widetilde{\Lambda} \subset \widetilde{\mathrm{Ein}}$ (Λ is simply connected). Then the space-like curve ∂H lifts to a curve, still spacelike, passing through the lifts of p and q to $\widetilde{\Lambda}$.

2 Anosov representations are quasi-Fuchsian

Let $P \subset \mathrm{SO}(2,n)$ be the stabilizer of an isotropic line. Then, $\mathrm{SO}(2,n)/P = \mathrm{Ein}^n$ is Einstein space. The open orbit $\mathcal{Y} \subset \mathrm{Ein}^n \times \mathrm{Ein}^n$ of the $\mathrm{SO}(2,n)$ action is the subset of pairs of points in Einstein space which are acausally related. These are exactly the pairs of endpoints of spacelike geodesics in AdS^{n+1}. We consider now P-Anosov (Merigot calls them $\mathcal{Y}$-Anosov) representations $\rho : \Gamma \to \mathrm{SO}(2,n)$. Let us recall the definition (Labourie). Consider the unit tangent bundle N of the compact hyperbolic manifold $\Gamma \backslash \mathbb{H}^n$, as well as the flat $\mathcal{Y}$-bundle E_ρ over N associated to ρ:

$$E_\rho = T^1\mathbb{H}^n \times_\rho \mathcal{Y}.$$

Denote the geodesic flow on N by ϕ^t; the flow lifts to E_ρ. Each factor of $\mathcal{Y}$ determines a sub-bundle of TE_ρ: Let E^s denote the span of the tangent directions to the stable sub-manifold of the geodesic flow on N and the tangent directions to the first Ein^n factor (see Spencer's notes). Let E^u denote the span of the tangent directions to the unstable manifold of the geodesic flow on N and the tangent directions to the second Ein^n factor. The tangent bundle of E_ρ splits as

$$TE_\rho = \Delta \oplus E^s \oplus E^u$$

where Δ is the line bundle tangent to the flow ϕ^t.

Then ρ is P-Anosov ($\mathcal{Y}$-Anosov) if there is a ϕ^t-invariant section $s : N \to E_\rho$ and there exists constants $a, b > 0$ such that:

- for any vector v in E^s over a point p of $s(T^1N)$, and for any $t > 0$:

$$\|d_p\phi^t(v)\| \le be^{-at}\|v\|$$

- for any vector v in E^u over a point p of $s(T^1N)$, and for any $t < 0$:

$$\|d_p\phi^t(v)\| \le be^{at}\|v\|$$

where $\|\cdot\|$ is some Riemannian metric (since Γ is cocompact, this definition is independent of the metric).

Both Merigot and Barbot use an alternate definition of Anosov (see Merigot Remark 5.4):

Remark 1 (Merigot Remark 5.4). *The Anosov section s is really two ρ-equivariant maps $\ell_\rho^+, \ell_\rho^- : T^1\mathbb{H}^n \to \mathrm{Ein}^n$. An equivalent formulation of the contraction properties is as follows: There exists a family of Riemannian metrics $g^{(x,v)}$ depending continuously on $(x,v) \in T^1\mathbb{H}^n$ and defined in a neighborhood of both $\ell_\rho^+(x,v)$ and $\ell_\rho^-(x,v)$ in Ein^n such that:*

- *The family is ρ-equivariant: Let $w \in T_{\ell_\rho^+(x,v)}$ or $w \in T_{\ell_\rho^-(x,v)}$. Then*

$$g^{\gamma(x,v)}(d\rho(\gamma)w, d\rho(\gamma)w) = g^{(x,v)}(w,w)$$

- *There exists $a, b > 0$, such that if $w \in T_{\ell_\rho^+(x,v)}$ and $t > 0$, then*

$$g^{\phi^t(x,v)}(w,w) \geq b^{-1}\exp(at)g^{(x,v)}(w,w)$$

  or if $w \in T_{\ell_\rho^-(x,v)}$ and $t > 0$ then

$$g^{\phi^t(x,v)}(w,w) \leq b\exp(at)g^{(x,v)}(w,w).$$

We use this second definition from now on, avoiding explicit reference to the metrics $g^{(x,v)}$ whenever the intuition is clear.

We now summarize Merigot's argument showing that Anosov representations (as above) are quasi-Fuchsian. This direction doesn't actually involve very much AdS geometry. In fact the following argument is very general, and is covered in Spencer's notes on the basics of Anosov representations. We repeat it here in this context.

The Anosov section s is really two maps $\ell_\rho^+, \ell_\rho^- : T^1\mathbb{H}^n \to \mathrm{Ein}^n$. Let $(x,v) \in N$ lie on the closed geodesic corresponding to $\gamma \in \Gamma$.

- Both $\ell_\rho^+(x,v), \ell_\rho^-(x,v)$ must be fixed points of $\rho(\gamma)$, and they must be distinct and acausally related. It follows that $\rho(\gamma)$ preserves the space-like geodesic in AdS^{n+1} with endpoints $\ell_\rho^+(x,v), \ell_\rho^-(x,v)$.
- It follows from the previous remark that $\ell_\rho^+(x,v)$ is an *attracting* fixed point of $\rho(\gamma)$ and that $\ell_\rho^-(x,v)$ is a *repelling* fixed point.
- A basic projective geometry argument implies that $\ell_\rho^+(x,v)$ (respectively $\ell_\rho^-(x,v)$) is the unique attracting (respectively repelling) fixed point of $\rho(\gamma)$.

Using these observations, Merigot proves

Proposition 2 (Merigot Prop 5.8). *Let $\alpha : T^1\mathbb{H}^n \to T^1\mathbb{H}^n$ be the map which flips the direction of tangent vectors: $\alpha(x,v) = (x,-v)$. Then $\ell_\rho^+ = \ell_\rho^- \circ \alpha$.*

The statement is true when (x, v) lies on a closed geodesic by the above observations. The proposition then follows by density of closed geodesics.

Now, define $\Lambda = \ell_\rho^+(N) = \ell_\rho^-(N)$. Since ℓ_ρ^+ (or ℓ_ρ^-) is flow invariant, we can really think of it as an equivariant map $\partial\mathbb{H}^n \to \mathrm{Ein}^n$ since the value only depends on the forward endpoint of the geodesic flow of (x, v). Thought of in this way, we argue that ℓ_ρ^+ is injective and its image is acausal. For consider two points (x, v) and (y, w) of $T^1\mathbb{H}^n$ with distinct (forward) endpoints on $\partial\mathbb{H}^n$. There is a third unit tangent vector (z, u) whose forward endpoint is the forward endpoint of (x, v) and whose backwards endpoint is the forward endpoint of (y, w). Then, using the proposition:

$$\ell_\rho^+(x, v) = \ell_\rho^+(z, u)$$
$$\ell_\rho^+(y, w) = \ell_\rho^+(z, -u) = \ell_\rho^-(z, u).$$

Since $(\ell_\rho^+(z, u), \ell_\rho^-(z, u)) \in \mathcal{Y}$ it follows that $\ell_\rho^+(x, v)$ and $\ell_\rho^+(y, w)$ are distinct and acausally related. This proves that Λ is an acausal embedded $n - 1$ sphere, and so ρ is quasi-Fuchsian.

3 Globally hyperbolic AdS space-times

Now to the converse result of Barbot, that quasi-Fuchsian representations are $\mathcal{Y}$-Anosov. We give only a vague summary. The strategy involves the geometry of globally hyperbolic AdS space-times.

Consider an acausal subset Λ of $\mathrm{Ein}^{n-1,1}$. Let $E(\Lambda)$ denote the points of $\mathrm{AdS}^{n,1}$ that can be joined to all points of Λ by a space-like geodesic. $E(\Lambda)$ is called the invisible set. It is open. Now, suppose $\rho : \Gamma \to \mathrm{SO}(2, n)$ is quasi-Fuchsian and Λ is the acausal $n - 1$ sphere preserved by the image of ρ. Then $M = \rho(\Gamma)\backslash E(\Lambda)$ is a globally hyperbolic maximal compact (GHMC) AdS manifold: It is globally hyperbolic with compact space-like level sets, and it is maximal with respect to inclusion (Mess). I won't get into the details here, but the assosiation $(\rho, \Lambda) \mapsto M$ is a bijection (Mess [Mes]).

3.1 The unit tangent bundle

Consider the space like unit tangent bundle $\mathcal{E}^1 E(\Lambda)$. Define the non-wandering set $\mathcal{N}(\Lambda)$ to be the subset of points (x, v) of $\mathcal{E}^1 E(\Lambda)$ such that the space-like geodesic passing though x tangent to v has both endpoints in Λ; in other words $\ell_+(x, v), \ell_-(x, v) \in \Lambda$. The non-wandering set is of course ρ invariant. The basic idea is that $\mathcal{N}(\Lambda)$ gives the connection between the dynamics of the geodesic flow on $\Gamma\backslash\mathbb{H}^n$ and the action of $\rho(\Gamma)$ on Λ. Specifically, Barbot proves:

Proposition 3 (Barbot Proposition 4.19)**.** *There is a ρ-equivairant homeomorphism $\mathfrak{f} : T^1\mathbb{H}^n \to \mathcal{N}(\Lambda)$ mapping orbits of the geodesic flow ϕ^t on $T^1\mathbb{H}^n$ to orbits of the space-like geodesic flow $\phi^t_{\mathcal{N}}$ on $\mathcal{N}(\Lambda)$.*

To prove this, the first step is to show the existence of a ρ-equivariant homeomorphism $j : \partial\mathbb{H}^n \to \Lambda$. This involves some AdS geometry: the group Γ is quasi-isometric to the convex hull $\mathrm{Conv}(\Lambda)$ of Λ in $\mathrm{AdS}^{n,1}$. Then both $T^1\mathbb{H}^n$ and $\mathcal{N}(\Lambda)$ are $\mathbb{R}$-bundles over $\partial\mathbb{H}^n \times \partial\mathbb{H}^n \setminus \mathcal{D} \cong \Lambda \times \Lambda \setminus \mathcal{D}$ (where $\mathcal{D}$ is the diagonal) and $j \times j$ gives an equivalence of the orbit spaces. Construction of $\mathfrak{f}$ is a standard exercise.

From the proposition, we get two flow invariant maps $\ell^+_\rho, \ell^-_\rho : T^1\mathbb{H}^n \to \Lambda$ defined by first applying $\mathfrak{f}$ and then following the geodesic flow in $\mathcal{N}(\Lambda)$ forwards and backwards to Λ. These are of course ρ-equivariant. The focus now is demonstrate the contraction properties of Remark 1 above. The metric $g^{(x,v)}$ at (near) $\ell^+_\rho(x, v)$ is defined as the visual metric from the point $x \in \mathrm{AdS}^{n,1}$. To make it Riemannian, we use *Wick rotation*. Specifically, there is a ρ-equivariant time-like vector field V defined in the convex core $\mathrm{Conv}(\Lambda)$ (which is the projection of $\mathcal{N}(\Lambda)$ to AdS); One may define a Riemannian metric on $\mathrm{Conv}(\Lambda)$ by using the AdS metric on the orthogonal to V and then flipping the sign of $\langle V, V\rangle$; projecting this metric (at x) onto the boundary Ein^n gives a Riemannian metric in a neighborhood of $\ell^+_\rho(x, v)$ (in fact defined on the entire space-like visual "sphere" of x, a copy of de Sitter space dS^n in Ein^n centered around x). Some fiddling is needed to make sure this is equivariant. Barbot shows that this metric satisfies the contracting properties. A key step is to show that for a sequence of group elements γ_n converging to a point in the Gromov boundary $\partial\Gamma$, the elements $\rho(\gamma_n) \in \mathrm{SO}(2, n)$ always have un-balanced distortion, meaning (most of) Einstein space is pushed toward one attracting fixed point.

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