Frame Field in $\mathbb{R}^3$: $X = \frac{\partial}{\partial x} - \frac{s}{2} y \frac{\partial}{\partial t}$ $\quad Y = \frac{\partial}{\partial y} + \frac{s}{2} x \frac{\partial}{\partial t}$ $\quad T = (1-s)\frac{\partial}{\partial t}$

Riemannian metric g so that these are orthonormal, That is, the inner product is given by sending vectors back by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{s}{2}y & \frac{s}{2}x & 1-s \end{bmatrix}^{-1} = \frac{1}{1-s}\begin{bmatrix} 1-s & 0 & 0 \\ 0 & 1-s & 0 \\ \frac{s}{2}y & -\frac{s}{2}x & 1 \end{bmatrix}$$

So the vector $\begin{bmatrix} u \\ v \\ w \end{bmatrix} = uX + vY + \frac{\frac{s}{2}(uy - vx)+w}{1-s} T$

Now, calculate the Levi-Civita connection ∇ for g. Since X, Y, T are ONB it reduces for

$$g(\nabla_A B, C) = \tfrac{1}{2}\left[g([A,B],C) + g([C,A],B) - g([B,C],A)\right]$$

Some Lie brackets: $[X,Y] = s\frac{\partial}{\partial t} = \frac{s}{1-s}T$ $\quad [X,T] = [Y,T] = 0.$

From this it is clear that $g(\nabla_A B, C) \neq 0$ only if all three fields are present.

$g(\nabla_Y X, T) = \frac{1}{2} g(\frac{-s}{1-s}T, T) = \frac{-s/2}{1-s}$ $\qquad g(\nabla_T X, Y) = \frac{1}{2}(-g([X,Y],T)) = \frac{-s/2}{1-s}$

$g(\nabla_X Y, T) = \frac{s/2}{1-s}$ $\qquad g(\nabla_T Y, X) = \frac{s/2}{1-s}$

$g(\nabla_X T, Y) = \frac{-s/2}{1-s}$ $\qquad g(\nabla_Y T, X) = \frac{s/2}{1-s}$

Represent this in a chart:

∇	X	Y	T
X	0	T	$-Y$
Y	$-T$	0	X
T	$-Y$	X	0

$\cdot \frac{s/2}{1-s}$

Now, the geodesic equation is $\nabla_{\gamma'}\gamma' = 0.$ Say $\gamma' = aX + bY + cT.$

Recall that $\nabla_\eta \xi$ is linear in η and $\nabla_\eta f\xi = \eta(f)\xi + f\nabla_\eta \xi$, and also additive in ξ

Furthermore in this case $\eta(f) = f' g(\eta, \gamma')$ if f is a function on $[0,1]$.

$$\begin{aligned}
\nabla_{aX+bY+cT}\, aX+bY+cT &= a\nabla_X aX + a\nabla_X bY + a\nabla_X cT \\
&\quad + b\nabla_Y aX + b\nabla_Y bY + b\nabla_Y cT \\
&\quad + c\nabla_T aX + c\nabla_T bY + c\nabla_T cT \\
&= a\left[g(X,\gamma')a'X + g(X,\gamma')b'Y + g(X,\gamma')c'T + b\tfrac{s/2}{1-s}T + c\tfrac{-s/2}{1-s}Y\right] \\
&\quad + b\left[g(Y,\gamma')a'X + g(Y,\gamma')b'Y + g(Y,\gamma')c'T + a\tfrac{-s/2}{1-s}T + c\tfrac{s/2}{1-s}X\right] \\
&\quad + c\left[g(T,\gamma')a'X + g(T,\gamma')b'Y + g(T,\gamma')c'T + a\tfrac{-s/2}{1-s}Y + b\tfrac{s/2}{1-s}X\right] \\
&= (a^2+b^2+c^2)(a'X + b'Y + c'T) + bc\tfrac{s}{1-s}X + ac\tfrac{-s}{1-s}Y \overset{\text{geodesic eq.}}{=} 0
\end{aligned}$$

Assuming $a^2+b^2+c^2=1$, this gives $\begin{cases} a' + bc\frac{s}{1-s} = 0 \\ b' + ac\frac{-s}{1-s} = 0 \\ c' = 0 \end{cases}$

Translating to $\gamma = (x,y,z)$, $\quad a = x'$, $b = y'$, $c = \frac{\frac{s}{2}(x'y - xy') + z'}{1-s}$ $\implies$ $a' = x''$, $b' = y''$, $c' = \frac{\frac{s}{2}(x''y - xy'') + z''}{1-s}$

So the geodesic equations are

$$\boxed{\begin{cases} x'' + y'\,\frac{\frac{s}{2}(x'y - xy') + z'}{1-s}\,\frac{s}{1-s} = 0 \\ y'' + x'\,\frac{\frac{s}{2}(x'y - xy') + z'}{1-s}\,\frac{-s}{1-s} = 0 \\ \frac{s}{2}(x''y - xy'') + z'' = 0 \end{cases}}$$