Continued Fractions on the Heisenberg Group

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The *Gauss map* on the interval $(0, 1)$ is given by

$$T(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor.$$

Applications of continued fractions

Continued fractions appear in:
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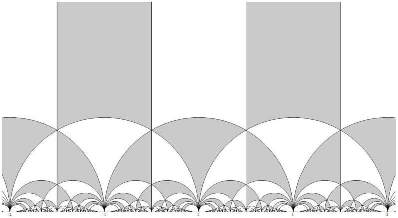

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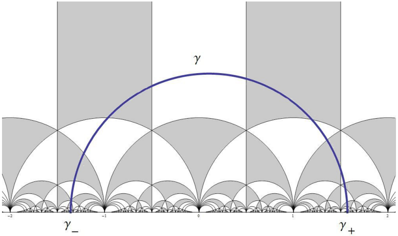

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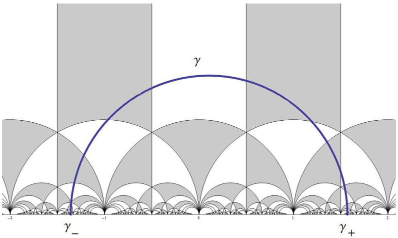

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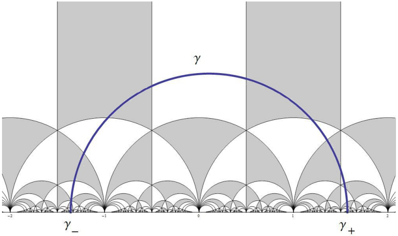

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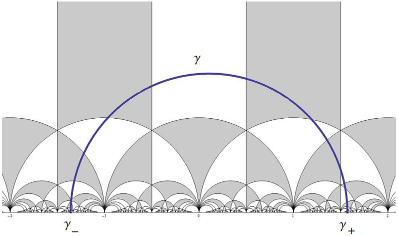

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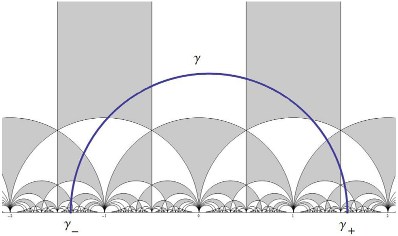

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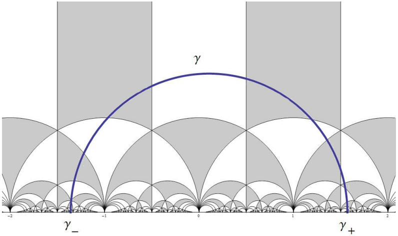

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Goal: replace $\mathbb{H}^2_{\mathbb{R}}$ with $\mathbb{H}^2_{\mathbb{C}}$, and $\mathbb{R}$ with the Heisenberg group $\mathcal{H}$.

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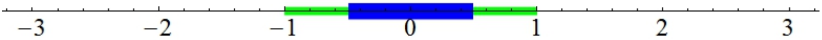

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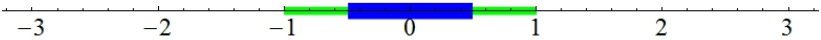
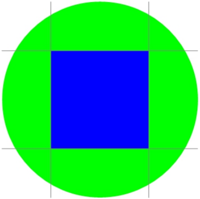


Complex continued fractions:



Heisenberg Group

The Heisenberg group $\mathcal{H}$ is $\mathbb{C} \times \mathbb{R}$ with group law

$$(z, t) * (z', t') = (z + z', t + t' + 2\mathrm{Im}\bar{z}z').$$

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Fix a fundamental domain K for $\mathcal{H}(\mathbb{Z})$, e.g.:

$$K_C = (-.5,.5] \times (-.5,.5] \times (-.5,.5]$$

$$K_D = \{p \in \mathcal{H} \mid d(p,0) \leq d(p,\gamma) \text{ for all } \gamma \in \mathcal{H}(\mathbb{Z})\}$$

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For $x \in \mathcal{H}$, define $\lfloor x \rfloor \in \mathcal{H}(\mathbb{Z})$ by the property $\lfloor x \rfloor^{-1} * x \in K$.

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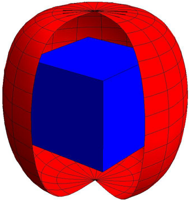

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$$\gamma_0 = \lfloor h \rfloor \qquad\qquad h_0 = \gamma_0^{-1} * h$$

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The convergents

$$\mathbb{K}\{\gamma_i\}_{i=0}^{n} = \left(\gamma_0 + \frac{1}{\cdots + \frac{1}{\gamma_n}}\right) = \gamma_0 \iota \gamma_1 \iota \cdots \iota \gamma_n.$$

The limit (if it exists):

$$\mathbb{K}\{\gamma_i\} = \lim_{n\to\infty} \mathbb{K}\{\gamma_i\}_{i=0}^{\infty}.$$

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Theorem (Pringsheim-style)

Let $\{\gamma_i\}_{i=0}^{\infty}$ be a sequence of elements of $\mathcal{H}(\mathbb{Z})$ satisfying $\|\gamma_i\| \geq 3$ for all i. Then the limit $\mathbb{K}\{\gamma_i\}$ exists.

Geodesic coding

The proof relies on an embedding $\mathcal{H} \hookrightarrow SU(2,1) = \mathrm{Isom}(\mathbb{H}^2_{\mathbb{C}})$ satisfying $\mathcal{H}(\mathbb{Z}) \hookrightarrow SU(2,1;\mathbb{Z}[i])$.

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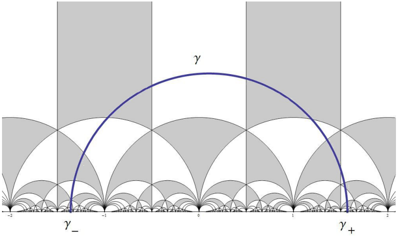

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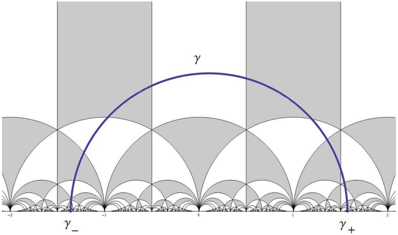

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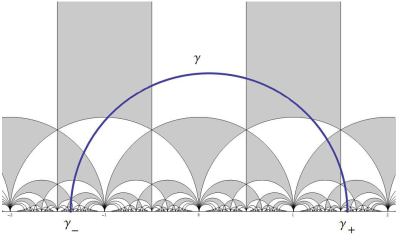

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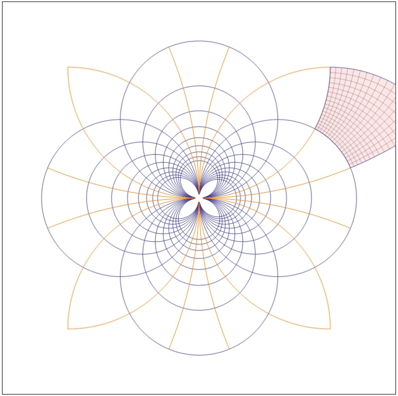

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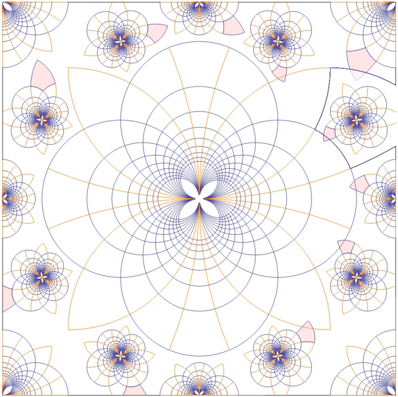

Invariant measures?

Numerical experiments suggest the following invariant measures for the Gauss map with respect to the fundamental domains K_C and K_D: