Lecture 8: Fillings In Nilpotent Groups

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Abstract

Previous lectures[1] discuss the general theory of Dehn and filling functions. We introduce nilpotent groups, discuss the connection between finitely generated and Lie groups, and state some results on filling functions of nilpotent groups.

1 Nilpotent Groups

1.1 Finitely Presented Groups

Definition 1.1. Given a group Γ, its *lower central series* is defined by commutators: $\Gamma_1 = \Gamma$ and $\Gamma_{i+1} = [\Gamma, \Gamma_i]$. Γ is called *nilpotent of class* c if $\Gamma_c \neq 1$ but $\Gamma_{c+1} = 1$.

Commutative groups are nilpotent of class 1. A class 2 nilpotent group is almost commutative: the commutator subgroup commutes with the full group. The analogy with commutative groups allows normalization arguments, as we demonstrate using growth rate.

Definition 1.2. Suppose Γ is generated by $g_1, \ldots, g_m \in \Gamma$. The *growth rate* of Γ is

$$B_\Gamma(N) = \#\{g \in G \mid l(g) \leq N\}, \tag{1.1}$$

where $l(g)$ is the length of the shortest word in $g_1, \ldots, g_m$ that represents g.

Definition 1.3. Given two functions $f, g : \mathbb{Z} \to \mathbb{Z}$, write $f \preccurlyeq g$ if for some $a, b, c, d \in \mathbb{Z}$ and all $N \in \mathbb{Z}$, $f(N) \leq g(aN + b) + cN + d$. Denote the induced equivalence relation by $\simeq$.

Let $\preccurlyeq$ and $\simeq$ also denote the analogous relations on functions from $\mathbb{R}$ to $\mathbb{R}$. We consider all functions up to $\simeq$ equivalence, without further mention.

It is clear that the definition of B_Γ is independent of the choice of generators.

Example 1.4. Suppose Γ is commutative and generated by $g_1, \ldots, g_m \in \Gamma$. Any $g \in \Gamma$ may be written as $g = g_1^{a_1} \cdots g_m^{a_m}$ for some $a_1, \ldots, a_m \in \mathbb{Z}$. The length of g is at most $\sum_i a_i$ (the normalization is not necessarily unique). If we require $l(g) \leq N$, then we have $a_i \leq N$, so a crude estimate gives $B_\Gamma(N) \leq N^m$. If the set of generators is minimal and torsion-free, then $\Gamma \cong \mathbb{Z}^m$, and $B_\Gamma(N) \simeq N^m$.

[1] These notes were written for the 2011 Summer School on Filling Invariants and Asymptotic Geometry. The notes for the other ten lectures are available at `http://mypage.iu.edu/~fisherdm/school11.html`.

Example 1.5. Suppose Γ is generated by $g_1, \ldots, g_m \in \Gamma$, and nilpotent of class 2. Because commutators commute with all the generators, we may again reduce words to a normal form. For example, we normalize $g = g_1 g_2 g_1 g_3^{-1}$:

$$g = g_1[g_1, g_2][g_1, g_2]^{-1} g_2 g_1 g_3^{-1} = g_1(g_1 g_2 g_1^{-1} g_2^{-1}) g_2 g_1 g_3^{-1} [g_1, g_2]^{-1} = g_1^2 g_2 g_3^{-1} [g_1, g_2]^{-1}.$$

More generally, any word of length N can be put in normal form in at most N^2 transpositions of adjacent letters. Suppose the commutator subgroup Γ_1 is generated by elements $h_1, \ldots, h_k$. Then a word of length N has normalized form

$$g = g_1^{a_1} \cdots g_m^{a_m} h_1^{b_1} \cdots h_k^{b_k}, \tag{1.2}$$

where $\sum_i a_i \leq N$. The normalization introduces at worst N^2 commutators. The commutators are then represented using $h_1, \ldots, h_k$, introducing a multiplicative term independent of N, so $\sum_i b_i \preccurlyeq N^2$. Combining these estimates, we have $B_\Gamma(N) \preccurlyeq N^{m+2k}$.

Example 1.5 can be turned into an inductive proof that a nilpotent group has polynomial growth. The term $m + 2k$ in the final estimate motivates the following definition:

Definition 1.6. Let Γ be a finitely generated nilpotent group. Then each quotient Γ_i/Γ_{i+1} is a finitely generated abelian group, so isomorphic to a direct sum of $\mathbb{Z}^{d_i}$ with a torsion group, for some $d_i \in \mathbb{N}$. Define the *homogeneous dimension* of Γ as $Q = \sum_i i d_i$.

A famous theorem links the growth rate to homogeneous dimension:

Theorem 1.7 (Bass [1])**.** *Let Γ be a finitely generated nilpotent group with homogeneous dimension Q. Then Γ has polynomial growth $B_\Gamma(N) \simeq N^Q$.*

A deep theorem of Gromov provided a converse:

Theorem 1.8 (Gromov [7])**.** *Let Γ be a finitely generated group with polynomial growth. Then Γ is virtually nilpotent.*

More recently, quantitative and more direct proofs have been provided by van den Dries - Wilkie, Kleiner, and most recently by Shalom-Tao [11] (see also Tao's blog).

1.2 Lie Groups

A *Lie group* G is a smooth manifold with a group structure whose product operation and inverse operation are smooth. The *Lie algebra* $\mathfrak{g}$ of G is the vector space of left-invariant vector fields on G. The Lie derivative of vector fields induces the *Lie bracket* on $\mathfrak{g}$ by

$$[\xi, \eta] = \mathcal{L}_X(Y) = \xi\eta - \eta\xi.$$

Alternately, on may identify the vector spaces $\mathfrak{g} = T_1 G$, where $T_1 G$ is the tangent space at the identity of G. In this case, one may speak of the exponential map $\exp : \mathfrak{g} \to G$.

Theorem 1.9. *Let G be a simply connected Lie group with Lie algebra $\mathfrak{g}$. Then the exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism.*

In particular, any simply connected Lie group is diffeomorphic to $\mathbb{R}^n$ for some n. Its group structure can be inferred from the Lie bracket on $\mathfrak{g}$ by means of the Campbell-Hausdorff formula (see [9]).

Definition 1.10. A Lie algebra $\mathfrak{g}$ is *nilpotent* if the lower central series $\mathfrak{g}_0 = \mathfrak{g}, \mathfrak{g}_{i+1} = [\mathfrak{g}, \mathfrak{g}_i]$ terminates with $\mathfrak{g}_c$ the last non-zero algebra.

Theorem 1.11. *A simply-connected Lie group is nilpotent if and only if its Lie algebra is nilpotent.*

1.3 Malcev Completion

Definition 1.12. Let Γ be a finitely generated nilpotent group. Elements $g_1, \ldots, g_m \in \Gamma$ are the *canonical generators* of Γ if every $g \in \Gamma$ may be written uniquely as

$$g = g_1^{a_1} \cdots g_m^{a_m} \tag{1.3}$$

for $a_1, \ldots, a_m \in \mathbb{Z}$. A group need not have canonical generators.

Note that canonical generators provide a bijection between Γ and $\mathbb{Z}^r$ that preserves some group structure. For example, the subgroup $\{g_{i_0}^{a_{i_0}} \cdots g_m^{a_m} \mid a_i \in \mathbb{Z}\}$ is normal in Γ.

Theorem 1.13 (Malcev [9])**.** *Let Γ be a finitely generated nilpotent group with no torsion elements. Then Γ has canonical generators.*

The key to Theorem 1.13 is to consider the abelian group $\Gamma/[\Gamma,\Gamma]$ and the lower-rank nilpotent group $[\Gamma,\Gamma]$. By an inductive assumption, both have canonical generators, and Malcev shows how to combine them into canonical generators for Γ.

Recall that a lattice in a Lie group G is a discrete subgroup Γ with G/Γ compact. Using canonical generators, Malcev proves the "Malcev completion" theorem:

Theorem 1.14 (Malcev [9])**.** *Let Γ be a finitely generated nilpotent group with no torsion elements. Then Γ is a lattice in a unique nilpotent Lie group G diffeomorphic to $\mathbb{R}^r$ for some r.*

The basic idea is to consider (1.3) and show that the group structure of Γ is given by polynomials in the a_i. These polynomials define a group structure on $G = \mathbb{R}^r$ by taking $a_i \in \mathbb{R}$ rather than $a_i \in \mathbb{Z}$. It is clear that G is a Lie group and $\Gamma \subset G$ is a lattice.

Conversely, multiplication in a simply connected nilpotent Lie group G is defined by polynomials whose coefficients are referred to as the *structural coefficients* of G. If the coefficients are rational, the Malcev completion may be reversed:

Theorem 1.15 (Malcev [9])**.** *Let G be a nilpotent Lie group with rational structural coefficients. Then G has a lattice.*

Example 1.16. The familiar example of Malcev completion is the extension $\mathbb{Z}^n \hookrightarrow \mathbb{R}^n$: an element of $\mathbb{Z}^n$ has the form $a_1e_1 + \ldots + a_ne_n$ for $a_i \in \mathbb{Z}$, and an element of $\mathbb{R}^n$ has $a_i \in \mathbb{R}$.

Example 1.17. The integer Heisenberg group $H_{\mathbb{Z}}^{2n+1}$ is defined as

$$H_{\mathbb{Z}}^{2n+1} = \{a_1, b_1, \ldots, a_n, b_n, t \mid [a_i, b_i] = t \text{ and all other brackets are trivial}\} \tag{1.4}$$

The Heisenberg group is nilpotent of class 2. As in Example 1.5, any word in the generators may be normalized to have the form

$$a_1^{c_1} b_1^{c_2} \cdots a_n^{c_{2n-1}} b_n^{c_{2n}} t^{c_{2n+1}}. \tag{1.5}$$

One then views $H_{\mathbb{Z}}^{2n+1}$ as the integer lattice within the real Heisenberg group

$$H_{\mathbb{R}}^{2n+1} = \mathbb{C}^n \times \mathbb{R} \text{ with } (z,t) * (z',t') = (z + z', t + t' + \operatorname{Im}\langle z, z'\rangle), \tag{1.6}$$

where $\langle \cdot, \cdot \rangle$ is the standard Hermitian pairing of complex vectors.

1.4 Carnot Groups

Definition 1.18. Suppose a simply connected nilpotent Lie group G has a Lie algebra $\mathfrak{g}$ with decomposition

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_c \tag{1.7}$$

such that the V_i are linear subspaces and furthermore $V_{i+j} = \operatorname{span}[V_i, V_j]$ for all i, j.[2] Then G is called a *Carnot group of homogeneous dimension* $Q = \sum_i i \dim V_i$.

Every Carnot group G is equipped with a one-parameter family of equivariant maps $\delta_r : G \to G$, $r \in \mathbb{R}$, such that the induced map on $\mathfrak{g}$ is

$$d\delta_r : (v_1, v_2, \ldots, v_c) \mapsto (rv_1, r^2 v_2, \ldots, r^c v_c). \tag{1.8}$$

Note in particular that δ_r commutes with the Lie bracket operation on $\mathfrak{g}$.

It is common to give a Carnot group a Riemannian metric by taking an inner product g on $\mathfrak{g}$ such that the V_i are orthogonal and extending g to a left-invariant tensor on G.

Various choices of the Riemannian metric g are bi-Lipschitz equivalent. However, sequences of such metrics may degenerate. In particular, one may desire a metric on which δ_r acts as a homothety.

Consider the metric $r\delta_{r*}g$, obtained by pulling back a δ_r-dilation of g and then rescaling it by r. By (1.8), $r\delta_{r*}g$ agrees with g on the subspace V_1, and more generally agrees with g up to a factor of r^{i-1} on the subspaces V_i. As $r \to \infty$, the sequence $r\delta_{r*}g$ degenerates to a *sub-Riemannian* metric:

Definition 1.19. Let G be a Carnot group and g_{CC} an inner product defined only on the V_1 subspace of $\mathfrak{g}$, and assumed infinite for all other pairings of vectors in $\mathfrak{g}$. The corresponding left-invariant tensor on G is called a *sub-Riemannian* or *Carnot-Carathéodory (CC)* inner product on G.

A path $\gamma : \mathbb{R} \to G$ is said to be *horizontal* if it almost everywhere lies along the vector fields in V_1: $\dot\gamma(t) \in V_1(\gamma(t))$. For $p, q \in G$, define the *CC metric* $d_{CC}(p, q)$ to be the infimal g_{CC}-length of horizontal paths between them.

Theorem 1.20 (Chow)**.** *Any two points in G are connected by a horizontal path. In particular, d_{CC} is a metric.*

The key idea in the proof (see [5]) is that the algebraic condition $\operatorname{span}[V_1, V_1] = V_2$ may be interpreted as the geometric statement "one may travel in a V_2 direction by attempting to make a loop along the V_1 subspace". In intuitive terms, one may travel vertically in G by going horizontally along a spiral staircase.

Recall that the Švarc-Milnor lemma ([3] p.140) states:

Theorem 1.21 (Švarc-Milnor)**.** *Let X be a length space. If Γ acts geometrically on X, then Γ is finitely generated and quasi-isometric to X.*

Thus, if Γ is a lattice in a Carnot group G, it is quasi-isometric to G with both the Riemannian and Carnot-Carathéodory metrics. Because the CC metric is homogeneous, its geometry corresponds to the large-scale geometry of the Riemannian metric on G and the word metric on Γ. In particular, the asymptotic cone of Γ is (G, d_{CC}).

[2] The space $[V_i, V_j]$ need not be a linear subspace of V_{i+j}. This is used in Lecture 9.

A vector not in V_1 may be represented by multiple Lie brackets, so the geodesics in G are not unique. Because a vector $\xi \in V_i$ is written as i iterated brackets of vectors in V_1, the corresponding directions are fractal: $(\exp_1 V_i, d_{CC})$ is bi-Lipschitz to $(\mathbb{R}^{\dim V_i}, \sqrt[i]{d_E})$, and the Hausdorff dimension of $\exp_1 V_i$ is $i \dim(V_i)$.

For a horizontal submanifold $M^k \subset G$, δ_r dilates the k-volume of M by factor r^k. In particular, d_{CC} is dilated by factor r. The Riemannian volume growth and the CC Hausdorff dimension of an arbitrary submanifold $M \subset G$ depends on the splitting of its tangent space, and can be computed as a weighted sum using (1.8). For example, if M^k has tangent directions strictly along V_3, then its Hausdorff dimension in the CC metric is $3k$. The dilation δ_r distorts its k-volume in the Riemannian metric by a factor of r^{3K}.

2 Filling Inequalities

2.1 Dehn Functions

Recall first the definition of a finitely presented group.

Definition 2.1. Let $g_1, \ldots, g_m$ be arbitrary symbols, and $\mathbb{F}$ the free group on $g_1, \ldots, g_m$. Let $r_1, \ldots, r_k \in \mathbb{F}$ and R the free group on $r_1, \ldots, r_k$. Because the r_i are elements of $\mathbb{F}$, there is a natural homomorphism $\iota : R \to \mathbb{F}$. Denote by $\overline{R}$ the normal closure of $\iota(R)$ in $\mathbb{F}$. If a group Γ is isomorphic to $\mathbb{F}/\overline{R}$, it is said to be a finitely presented group with presentation $\langle g_1, \ldots, g_m \mid r_1, \ldots, r_k \rangle$.

A basic invariant of a finitely presented group is its Dehn function:

Definition 2.2. Let $w \in \overline{R}$. Define $|w|$ to be the length of w as an element of $\mathbb{F}$, and the Dehn function $\delta_\gamma(N)$ as:

$$\begin{aligned} \delta_\Gamma(w) &= \text{minimum length of } W \text{ as a word in conjugates of } r_1, \ldots, r_k, \\ \delta_\Gamma(N) &= \min\{\delta_\Gamma(w) \mid w \in \overline{R}, |w| \leq N\}. \end{aligned}$$

As with the growth function, different choices of presentation yield different Dehn functions. However, it is clear that δ_Γ is well-defined up to the equivalence $\simeq$ from Definition 1.3.

Recall that if Γ is finitely presented, it acts geometrically on an associated "Cayley complex":

Definition 2.3. Let Γ be a finitely presented group. The *Cayley complex* of Γ is a cellular 2-complex whose points are the elements of Γ, edges are conjugates of the generators g_i, and faces are conjugates of the relators r_i.

A word $w \in \mathbb{F}$ may be thought of as a sequence of edges in the Cayley complex of Γ starting at the identity. The sequence of edges forms a loop exactly if $w \in \overline{R}$. It may be contracted back to the identity along the faces of the Cayley complex, and $\delta_\Gamma(w)$ gives the minimal number of faces that need to be crossed.

Interest in Dehn functions is primarily stimulated by the following deep result of Gromov:

Theorem 2.4 (Gromov [8])**.** *Let Γ be finitely presented group. Then $\delta_\Gamma(N) \simeq N$ exactly if Γ is a hyperbolic group.*

We will next discuss certain results concerning the Dehn functions of nilpotent groups.

2.2 Free Nilpotent Groups and General Bounds on Dehn Functions

Definition 2.5. Let $\mathbb{F} = \langle g_1, \ldots, g_n \rangle$ be a free group on n generators. Let F_i be the i^{th} term of the lower central series for $\mathbb{F}$ and define $\mathbb{F}_c = \mathbb{F}/\overline{F_{c+1}}$ to be the *free nilpotent group of class c on n generators.*

We are interested in computing the Dehn function $\delta_{\mathbb{F}_c}$. Recall from Example 1.5 that if we were interested in the growth function, we would take a word in $\mathbb{F}$, say $w = g_1g_2g_1g_3^{-1}$, and put it in normal form like $g_1^2g_2g_3^{-1}[g_1, g_2]^{-1}$. Finding a normal form for all elements gives an upper bound on the number of words of a given length.

In the case of the Dehn function, we are interested writing a word w representing the identity as

$$w = a_1r_1^{\epsilon_1}a_1^{-1}a_2r_2^{\epsilon_2}a_2^{-1}\cdots a_lr_l^{\epsilon_l}a_l^{-1}, \tag{2.1}$$

where $a_i \in \mathbb{F}, \epsilon_i = \pm 1, r_i \in F_{c+1}, l \in \mathbb{N}$. We then ask what the smallest l_0 is such that every w of length less than N may be written in the form (2.1) for $l \leq l_0$.

Unfortunately, we can't normalize (2.1) as is: we are working in $\mathbb{F}$, which has trivial center.

Definition 2.6. The centralized Dehn function of $\Gamma = \langle g_1, \ldots, g_m \mid r_1, \ldots, r_k \rangle$ is given by

$$\begin{aligned}
\delta_\Gamma^{\text{cent}}(\omega) &= \min\{l \mid w \in a_1r_1^{\epsilon_1}a_1^{-1}\cdots a_lr_l^{\epsilon_l}a_l^{-1}[R, \mathbb{F}] \text{ for some choice of } a_i, r_i, \epsilon_i\},\\
\delta_\Gamma^{\text{cent}}(N) &= \min\{\delta_\Gamma^{\text{cent}}(\omega) \mid \omega \in R, |\omega| \leq N\}.
\end{aligned}$$

Clearly, $\delta_\Gamma^{\text{cent}} \leq \delta_\Gamma$ and the equivalence class $\delta_\Gamma^{\text{cent}}$ is independent of the generating set of Γ. Considering cosets makes normalization arguments like that in Example 1.5 possible, and a combinatorial argument shows:

Theorem 2.7 (Baumslag-Miller-Short [2])**.** *Let $\mathbb{F}_c$ be a free nilpotent group of class c and finite rank at least 2. Then $N^{c+1} \preccurlyeq \delta_{\mathbb{F}_c}^{cent}(\preccurlyeq \delta_{\mathbb{F}_c})$.*

Using the action of $\mathbb{F}_c$ on its Cayley complex, Bridson-Pittet in [4] showed that the above estimate is in fact sharp: $\delta_{\mathbb{F}_c}(N) \simeq N^{c+1}$. Furthermore, the following was proven more recently:

Theorem 2.8 (Gersten-Holt-Riley[6])**.** *Let Γ be a finitely presented nilpotent group of class c. Then $\delta_\Gamma(N) \preccurlyeq N^{c+1}$.*

Perhaps counter-intuitively, free nilpotent groups realize this bound. As we show below, other groups do not.

The centralized Dehn function computes filling volume in $\overline{R}/[\overline{R}, \mathbb{F}]$. In fact, considering the long exact homology sequence associated with the presentation $\Gamma = \mathbb{F}/\overline{R}$ shows that $\overline{R}/[\overline{R}, F]$ splits as

$$\overline{R}/[\overline{R}, F] \cong H_2\Gamma \oplus \mathbb{Z}^l \tag{2.2}$$

for some l, where $H_2\Gamma$ is the second integer homology of Γ. It turns out that the $\mathbb{Z}^l$ term contributes little to δ^{cent}, and one can use (2.2) to compute the δ_{cent} for groups with known $H_2\Gamma$. For example, Baumslag, et al. show:

Theorem 2.9 (Baumslag-Miller-Short [2])**.** *Let Γ be a finitely presented group with finite $H_2\Gamma$. Then $\delta_\Gamma^{cent}(N) \simeq N$.*

2.3 Central Products and Non-Polynomial Dehn Functions

Given Theorem 2.8 and the results for free nilpotent groups, one might conjecture that the Dehn function of a nilpotent group of class c is N^{c+1}, or at least polynomial. Below, we consider class-2 nilpotent Lie groups and show that this is not the case.

According to Theorem 2.8, a class-2 nilpotent group has at most cubic Dehn function. In [10], Olshanskii and Sapir provide a combinatorial proof that the higher Heisenberg groups have, in fact, a quadratic Dehn function.

Theorem 2.10 (Olshanskii-Sapir [10])**.** *For $n \geq 2$, the integer Heisenberg group $H^{2n+1}_{\mathbb{Z}}$ has quadratic Dehn function.*

Furthermore, they claim their proof can be adjusted to prove the following result, whose proof appears in [13]:

Theorem 2.11. *Let Γ be a a nilpotent finitely presented group of class 2 with quadratic Dehn function. Denote by Γ^2 the class-2 nilpotent group $\Gamma \times \Gamma / \sim$, with $(g,1) \sim (1,g)$ for $g \in [\Gamma, \Gamma]$. Then $\delta_{\Gamma^2}(N) \preccurlyeq N^2 \log(N)$.*

We have thus demonstrated groups with Dehn functions N^{c+1} and N^2. We have provided a bound of $N^2 \log(N)$ for another group, but have not claimed that it is sharp. The next lecture (Lecture 9) discusses the following result:

Theorem 2.12 (Wenger [12])**.** *There exists a nilpotent group with non-polynomial Dehn function.*

The key idea in the construction is that quotients of nilpotent Lie groups by central subgroups not generated by commutators cannot have quadratic Dehn function.

2.4 Lipschitz Chains and Filling Volume

We now define Dehn and filling functions for Lie groups, and provide some preliminary definitions.

Definition 2.13. Let M be a Riemannian manifold and Σ_k the standard k-simplex. A singular Lipschitz k-chain f is a formal finite sum $f = \sum_i a_i f_i$ where $a_i \in \mathbb{R}$ and $f_i : \Sigma_k \to M$ are Lipschitz maps. Note that the boundary of a Lipschitz chain, defined piecewise, is a Lipschitz chain.

To define the mass of a Lipschitz chain, we refer to the following standard result, which also applies in the Riemannian setting:

Theorem 2.14 (Rademacher's Theorem)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz. Then f is almost everywhere differentiable.*

Definition 2.15. Let f be a Lipschitz chain, as above. The *mass* of f is the weighted sum

$$\operatorname{mass}_k(f) = \sum_i |a_i| \int_{B^k} |Jf_i| \, dV. \tag{2.3}$$

There are two natural filling functions for a Riemannian manifold:

Definition 2.16. Suppose $f : S^i \to M$ is Lipschitz. Define the i^{th} Dehn function of M by:

$$\delta^i(f) = \inf\{\text{mass}_{i+1}(g : B^{i+1} \to M) \mid g \text{ is Lipschitz and } \partial g = f\},$$
$$\delta^i(V) = \sup\{FV^i(f) \mid f : S^i \to M \text{ is Lipschitz and } \text{mass}_i(f) \leq V\}.$$

Definition 2.17. Suppose f is a singular Lipschitz $i+1$-chain in M. Define the filling volume in M as:

$$FV^{i+1}(f) = \inf\{\text{mass}_{i+1}(g) \mid g \text{ is a singular Lipschitz } i+1\text{-chain and } \partial g = f\},$$
$$FV^{i+1}(V) = \sup\{FV^{i+1}(f) \mid f \text{ is a singular Lipschitz } i\text{-chain and } \text{mass}_i(f) \leq V\}.$$

The equivalence classes of δ^i and FV^{i+1} do not depend on the choice of the Riemannian metric. Unfortunately, the two functions do not always agree for small i. See Lecture 5 or [13] for the relationship between them.

The Federer-Fleming Decomposition Theorem is the main tool for analyzing the filling functions in manifolds. We state the variation of the theorem found in [13]:

Theorem 2.18 (Federer-Fleming)**.** *Suppose M is a triangulated manifold and f is a singular Lipschitz k-chain of mass l. Then there exists a simplicial k-chain f' and a singular Lipschitz $k+1$-chain g such that:*

1. $\partial g = f' - f$,
2. $mass_k(f') \leq cl$,
3. $mass_{k+1}(g) \leq cl$,

where c depends only on M and its triangulation.

Suppose a finitely generated group Γ acts geometrically on a manifold M. By Theorem 2.18, we have that $\delta_\Gamma \simeq \delta^1(M)$. In particular, the Dehn function of a torsion-free nilpotent group agrees with the first Dehn function of its completion.

2.5 Filling Volume Bounds

In this section, we discuss the following theorem, which in particular provides an example of Lie groups with quadratic Dehn functions :

Theorem 2.19 (Young [13])**.** *Let $c, k > 0$. The jet group $J^m(\mathbb{R}^k)$, when endowed with a left-invariant Riemannian metric, has*

$$FV^i(V) \simeq \delta^{i-1}(V) \simeq V^{\frac{i}{i-1}}, \tag{2.4}$$

where $i \leq k$, and

$$FV^{k+1}(V) \simeq \delta^k(V) \simeq V^{\frac{k+m+1}{k}}. \tag{2.5}$$

The jet groups $J^m(\mathbb{R}^k)$ are defined below. These are class-m Carnot groups that include the Heisenberg groups $H^{2k+1}_{\mathbb{Z}} = J^1(\mathbb{R}^k)$. Jet groups have lattices, so we have, in contrast to the bound $\delta_\Gamma(N) \preccurlyeq N^{c+1}$ of Theorem 2.8,

Corollary 2.20. *Let $c > 0$. Then there exists a finitely presented nilpotent group of class c such that δ_Γ is quadratic.*

The proof of Theorem 2.19 consists of two parts: the construction of well-behaved triangulations of the jet groups, and a procedure for filling cycles in such spaces. We first compute $\delta^1(\mathbb{R}^2)$ to provide an example that demonstrates the filling method.

Example 2.21. Suppose $f : S^1 \to \mathbb{R}^2$ is a loop in $\mathbb{R}^2$ of length $l > 1$. Approximate f by a path f_i along the edges of the integer lattice. The length of f_i is approximately l. Repeat, replacing the integer grid with $2^i\mathbb{Z}^2$ to obtain an approximation f_i. For i_0 with $2^{i_0} \gg l$, we have $f_{i_0} = 0$.

Next, consider the difference between f_i and f_{i+1}. The first approximation f_i is composed of roughly $l2^{-i}$ segments of length 2^i, and is connected to f_{i+1} by an interpolating "annulus" g_i made up of roughly $l2^{-i}$ squares of area 2^{2i}. More precisely, we have $\partial g_i = f_i - f_{i+1}$, and $\mathrm{mass}_2(g_i)$ is roughly $l2^i$. We then have:

$$\partial(g_1 + \ldots + g_{i_0}) = f$$
$$\mathrm{mass}_2(g_1 + \ldots + g_{i_0}) \approx \sum_{i=1}^{i_0} l2^i \leq l2^{i_0} \approx l^2.$$

The fact that the grids $2^i\mathbb{Z}^2$ are related by homotheties is essential, as it provides a uniform way to approximate f and relate the approximations f_i.

Carnot groups provide the natural setting for generalizing Example 2.21. In particular, one works with the jet groups $J^m(\mathbb{R}^k)$, which have a large horizontal subspace.

Definition 2.22. A Lipschitz triangulation τ of a Carnot group G is k-horizontal if every k-simplex of the triangulation is embedded horizontally in G.

Note that the k-simplices of a horizontal triangulation rescale by a factor of r under the map $\delta_r : G \to G$. Given a horizontal triangulation, Young uses the Federer-Fleming Theorem 2.18 to approximate a cycle f by simplicial cycles f_i at various scales, and relates the approximations f_i and f_{i+1} in a uniform way.

Definition 2.23. Let $k \geq 2$. The gradient of a function $f : \mathbb{R}^k \to \mathbb{R}$ can be interpreted as a section of the cotangent bundle of $\mathbb{R}^K$ given by $df : \mathbb{R}^k \to T^*\mathbb{R}^k$. Similarly, derivatives of order up to m are sections of the *jet bundle* $J^m(\mathbb{R}^k) = \mathbb{R}^k \times W$, where

$$W = \mathbb{R} \times (\mathbb{R}^*)^k \times S^2((\mathbb{R}^*)^k) \times \cdots \times S^m((\mathbb{R}^*)^k)$$

and S^i denotes the i^{th} symmetric power. A function $f : \mathbb{R}^k \to \mathbb{R}$ has a *prolongation* $j^m f : \mathbb{R}^n \to J^m(\mathbb{R}^n)$ defined at each point $p \in \mathbb{R}^k$ by placing the i^{th} partials of f at p in the $S^i((\mathbb{R}^*)^k)$ component of W.

Consider a point $p = (x, x') \in J^m(\mathbb{R}^n)$. There is a unique polynomial f_p in k variables and of order m such that p is in the image of $j^m f_p$. That is, $j^k(f_p)(x) = x'$. The differential data x' may be transferred to lie over another point $y \in \mathbb{R}^n$ as $j^m f_p(y)$, the partials of f_p over y.

Provide $J^m(\mathbb{R}^n)$ with a group structure by taking

$$(x, x') * (y, y') = (x + y, j^m f_p(y) + y') \tag{2.6}$$

With this group structure, $J^m(\mathbb{R}^n)$ is a Carnot group of class $m+1$. As Young demonstrates, the structural constants of $J^m(\mathbb{R}^n)$ are rational, so $J^m(\mathbb{R}^n)$ contains lattices.

Example 2.24. The Heisenberg group $H_{\mathbb{R}}^{2n+1}$ is isomorphic to $J^1(\mathbb{R}^n)$.

Young uses the key fact that prolongations of smooth maps are horizontal to construct m-horizontal triangulations of the jet groups. Since n may be taken arbitrarily high without affecting the nilpotence class of the jet group, they may be used to study filling volume in arbitrarily high dimension.

2.6 Dehn Functions of Lie Group Quotients

The following theorem deals with the first Dehn function δ^1 of a Carnot group G. However, the results may be translated to a statement about a lattice $\Gamma \subset G$. Note that a general Carnot group need not have a lattice.

Theorem 2.25 (Young [13])**.** *Let G be a class-2 nilpotent Lie group with Lie algebra $\mathfrak{g}$ and a quadratic Dehn function δ^1. Suppose $W \subset \mathfrak{g}$ is a subalgebra generated by elements of the form $[\xi, \eta]$, and G' is a Lie group with Lie algebra $\mathfrak{g}/W$. Then G' has a quadratic Dehn function δ^1.*

The proof of Theorem 2.25 focuses on the geometric interpretation of the commutator.

More specifically, note that $\mathfrak{g}$ decomposes as $V_1 \oplus V_2$ and suppose W is generated by a single commutator $[w_1, w_2]$ with $w_1, w_2 \in V_1$. Then a loop γ of length l in G' lifts to a path $\tilde{\gamma}$ in G between 0 and $\exp \alpha[w_1, w_2]$ for some α. We may assume $\alpha > 0$, and Young shows $\alpha \simeq l^2$.

Let $\tilde{\eta}$ be a path in G corresponding to exponentiating, in order, the vectors $\sqrt{\alpha}w_1$,$\sqrt{\alpha}w_2$, $-\sqrt{\alpha}w_1$, $-\sqrt{\alpha}w_2$. This curve has length $4\sqrt{\alpha}$ and connects the same endpoints as γ. Furthermore, it corresponds to a loop η in G' that may be filled by a square of area α.

Because G has a quadratic Dehn function, $\tilde{\gamma}$ may be homotoped to $\tilde{\eta}$ using a homotopy of area comparable to l^2. This may be projected to G' to homotope γ to η, which is then closed off using the square of length approximately l^2.

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