Bi-Lipschitz extension from boundaries of certain hyperbolic spaces

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Outline

1. Quasi-conformal extension
2. Bi-Lipschitz extension
3. Metric similarity spaces
4. Boundaries of hyperbolic spaces

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Conformal extension

Definition

A differentiable function $f : \mathbb{R}^n \to \mathbb{R}^n$ is *conformal* if $df|_p \in O(n)$ for all p.

Theorem

Every conformal map of S^{n-1} is a Möbius map and extends to a conformal map of $\overline{B^n}$. Conversely, every conformal map of B^n extends to a map of $\overline{B^n}$, conformal on S^{n-1}.

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Quasi-conformal extension

Definition

A differentiable function $f : \mathbb{R}^n \to \mathbb{R}^n$ is *k-quasi-conformal* if for almost all p $df|_p$ satisfies

$$\left| \frac{\lambda_{\max}(p)}{\lambda_{\min}(p)} \right| < k$$

where $\lambda_{\max}(p)$ and $\lambda_{\min}(p)$ are the eigenvalues of $df|_p$ of maximum and minimum norm.

Theorem (Gehring)

Every quasi-conformal map of B^n extends to a map of $\overline{B^n}$, quasi-conformal on S^{n-1}.

Theorem (Beurling-Ahlfors, Ahlfors, Carleson, Tukia-Väisälä)

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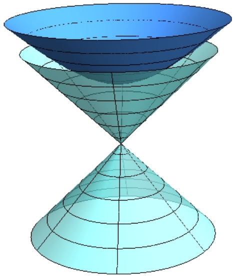

Definition

Real hyperbolic space $\mathcal{H}^n_{\mathbb{R}}$ is the ball B^n with a Riemannian metric invariant under the group $O(n,1)$.

Definition

A function $f : X \to Y$ is L-bi-Lipschitz if for all $x, x' \in X$

$$L^{-1} d_X(x,x') \leq d_Y(fx, fx') \leq L d_X(x,x')$$

Theorem (Beurling-Ahlfors, Ahlfors, Carleson, Tukia-Väisälä)

Let $f : S^{n-1} \to S^{n-1}$ be quasi-conformal. Then f extends to a map on B^n, bi-Lipschitz with respect to the hyperbolic metric.

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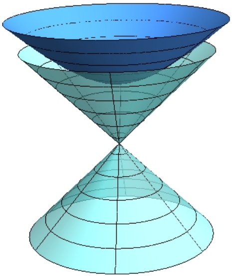

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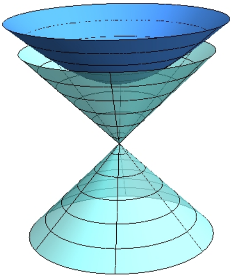

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Quasi-symmetry

Question

Let d_S be a metric on S^{n-1} and d_B a metric on B^n.
Let $f : S^{n-1} \to S^{n-1}$ be quasi-conformal with respect to d_S.
Does f extend continuously to $F : \overline{B^n} \to \overline{B^n}$ so that $F|_{B^n}$ is bi-Lipschitz with respect to d_B?

Definition

Let X, Y be metric spaces and $f : X \to Y$ a homeomorphism.
Then f is an η-quasi-symmetry if for all $x, x', x'' \in X$

$$\frac{d_Y(fx, fx')}{d_Y(fx, fx'')} \leq \eta\left(\frac{d_X(x, x')}{d_X(x, x'')}\right),$$

where $\eta : [0, \infty) \to [0, \infty)$ is a homeomorphism.

Remark

Quasi-symmery generalizes quasi-conformality to metric spaces.

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Metric similarity spaces

Definition

Let (H, d_H) be a metric space homeomorphic to $\mathbb{R}^{n-1}$. A *stacked tiling* on H is $(\Gamma, \Gamma', \alpha, K)$ satisfying:

1. $\Gamma \subset \mathrm{Isom}(H)$ is a discrete co-compact subgroup.
2. α is a dilation by some $a > 1$.
3. $\alpha\Gamma\alpha^{-1} \subset \Gamma$,
4. K is a fundamental domain for Γ.
5. $\alpha K = \cup_{\gamma\in\Gamma'} K$ for some finite $\Gamma' \subset \Gamma$.

Definition

A *metric similarity space* H^+ with *base* H is $H \times \mathbb{R}^+$ with a complete Riemannian metric so that:

1. Γ acts by isometries in the first factor.
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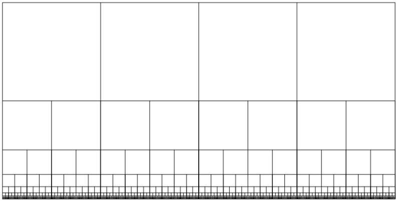
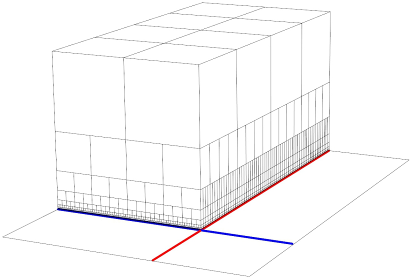


$H =$ Euclidean space $\mathbb{R}^{n-1}$
$H^+ = \mathcal{H}^n_{\mathbb{R}}$
$\Gamma = \mathbb{Z}^{n-1}$



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Theorem (L.)

Let H^+ be a metric similarity space of dimension $n \neq 4$ with base H. Every quasi-symmetry of H extends continuously to a bi-Lipschitz map of H^+. Conversely, every bi-Lipschitz map of H^+ extends to a quasi-symmetry of H.

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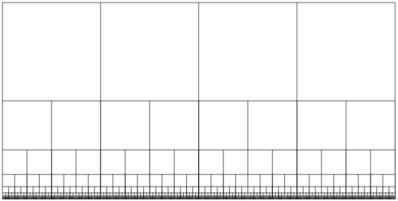

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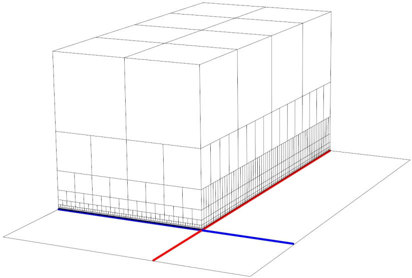

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Proof sketch: extension to H^+

1. Start with $f : H \to H$ quasi-symmetric.
2. Naïve extension $\widetilde{f}$

$$\widetilde{f}(z,t) = \left(f(z,t), \sup_{|z-w|=t} |f(z) - f(w)| \right)$$

3. Tiling of H^+
4. Piecewise-linear approximation of $\widetilde{f}$ on disjoint tiles
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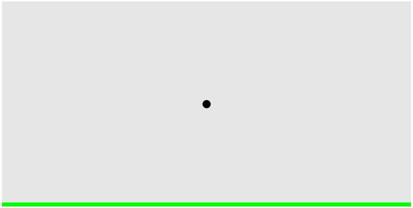

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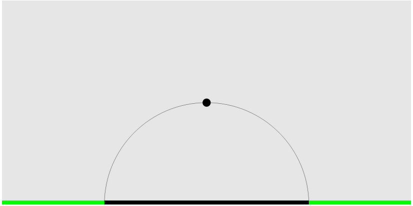

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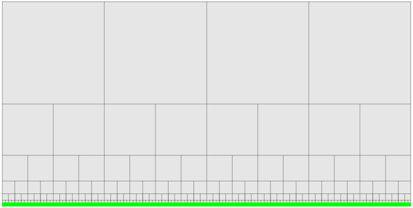

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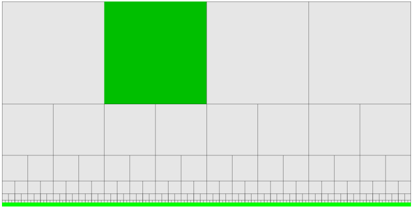

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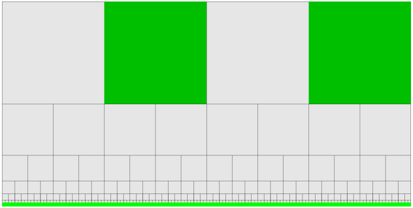

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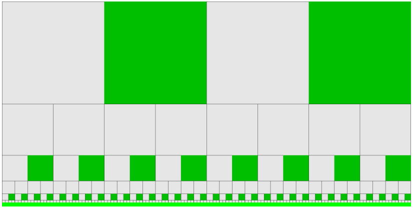

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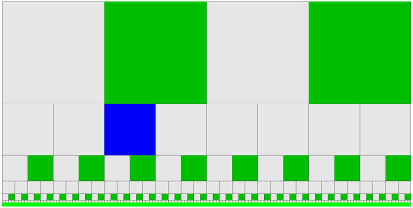

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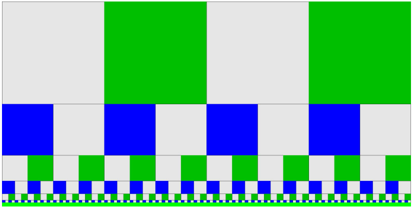

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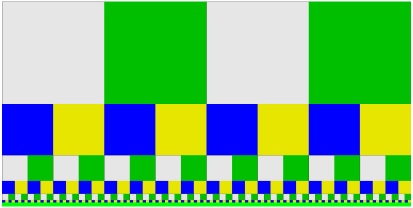

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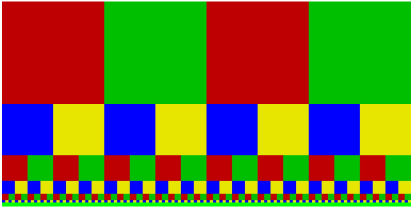

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Gromov hyperbolic spaces

Definition

Let X be a geodesic metric space, denoting geodesic segments by $[\cdot,\cdot]$, and let $\delta \geq 0$. X is (δ-)hyperbolic if for all $x, x', x'' \in X$,

$$[x, x''] \subset N_\delta([x, x'] \cup [x', x'']).$$

Definition

Let X be a hyperbolic space. Define the boundary of X by

$$\partial X = \{\text{geodesic rays in } X\}/\text{finite Hausdorff distance.}$$

Let $0 \in X$ and write $(\xi|\eta)_0 = d(0, [\xi, \eta])$.
A metric on X is *a-visual* if it is quasi-symmetric to the expression

$$d_{0,a}(\xi, \eta) = e^{-a(\xi|\eta)_0}.$$

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Proof sketch: extension to H

Theorem (Bonk-Schramm)

Let (X, d_X) be a complete, bounded metric space of diameter D. Then there is a hyperbolic metric on $Con(X) = X \times (0, D]$ so that d_X is a visual metric on $\partial Con(X)$. Every quasi-isometry of $Con(X)$ extends to a quasi-symmetry of X.

Theorem (L.)

Let H^+ be a metric similarity space with base H. Then H^+ is quasi-isometric to $Con(H)$.

Proof sketch.

Adjacency graph for the tiling. □

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