Math 675 Potential Midterm Exam Problems

The exam will consist of 10 problems, separated into definitions, examples, and proofs. Here are some potential questions.

The list of questions may be modified before the actual exam is compiled. You should think about what additional problems might be added.

Definitions

1. Metric
2. Closure
3. Vector space
4. Contact point
5. Cauchy
6. Complete
7. Sequentially compact
8. Compact
9. Separable
10. Open
11. Closed
12. Continuous
13. Homeomorphism
14. Isometry
15. Completion
16. Contraction
17. Linearly independent
18. Basis
19. Span
20. Linear functional
21. Null space
22. Hyperplane
23. Codimension
24. Convex
25. Convex body
26. Convex functional
27. Quotient space
28. Minkowski functional
29. Subspace
30. d_p
31. $C_p[a, b]$
32. ℓ_p

Proofs

1. For $a < b$ in $\mathbb{R}$, the closure of (a, b) is $[a, b]$.
2. In $\mathbb{R}^n$, the metric d_∞ is the limit of the functions d_p as $p \to \infty$.
3. The spaces $C_\infty[a, b]$ and $C_\infty[c, d]$ are isometric for any $a < b$ and $c < d$.
4. A point $x \in X$ is a limit point of A if and only if for every $\epsilon > 0$ the ball $B_\epsilon(x)$ contains a point of A not equal to x.
5. The closure of A equals the union of the set of isolated points A with the set of limit points of A.
6. What is the closure of $\mathbb{Q} \subset \mathbb{R}$? Prove that you are correct.
7. The following are equivalent for a function $f : X \to Y$ between two metric spaces:
   (i) f is continuous,
   (ii) $f^{-1}(A)$ is an open set for every open set $A \subset Y$,
   (iii) $f^{-1}(B)$ is a closed set for every closed set $B \subset Y$.
8. The set of invertible 2-by-2 matrices is open. (Interpret the space of all 2-by-2 matrices as $\mathbb{R}^4$ with the usual topology.) You may use your favorite definition of continuity.
9. The set of determinant-one 2-by-2 matrices is closed. You may use your favorite definition of continuity.
10. Let A be a linearly independent set in a metric vector space V. Assuming that V is finite-dimensional, prove that the set $\mathbb{R}A = \{\lambda a : \lambda \in \mathbb{R}, a \in A\}$ is closed.
11. Suppose two metric spaces X and Y are bi-Lipschitz equivalent. Prove that that X is complete if and only if Y is complete.
12. The image of a compact set under a continuous function is compact.
13. Let $f : X \to X$ be a contraction, and $Y \supset X$ a completion of X. Prove that f extends to a contraction of Y. Use this to conclude that there is no homeomorphism of $(0, 1)$ that is also a contraction.
14. The contraction theorem requires a mapping that is L-Lipschitz for $L < 1$. Prove that the result is sharp. That is, allowing $L = 1$ in the theorem would make it false.
15. A closed subset of a compact space is compact.
16. Suppose f, f_1, f_2 are linear functionals such that $f(x) = 0$ if $f_1(x) = 0$ and $f_2(x) = 0$. Prove that there are constants a_1 and a_2 such that $f = a_1 f_1 + a_2 f_2$.
17. If X is compact and $f : X \to \mathbb{R}$ is continuous, then f attains a maximum value.

18. If $A, B \subset X$ are compact subsets of a metric space X, then $A \cup B$ is compact.

Examples

1. A bounded set that is not compact.
2. A metric vector space that is not separable.
3. A linearly independent set A in a metric vector space V such that the set $\mathbb{R}A = \{\lambda a : \lambda \in \mathbb{R}, a \in A\}$ is not closed.
4. A union of closed sets that is not closed.
5. An intersection of open sets that is not open.
6. Homeomorphic metric spaces X and Y such that X is complete but Y isn't.
7. A compact set C and a continuous function f such that $f^{-1}(C)$ is not compact.
8. A contraction that does not have a fixed point.
9. Let V be a non-trivial vector space. Give an example of a non-trivial functional on V^*.
10. A convex set that is not a convex body.
11. Two sets that cannot be separated by a linear functional.
12. A Cauchy sequence in $C_2[a, b]$ that does not converge to a function in $C_2[a, b]$.